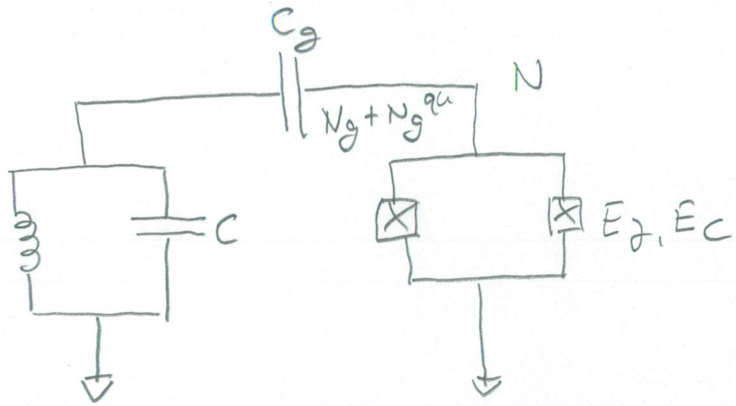
Jaynes - Cummings Hamiltonian in Circuit QED



$$\hat{H} = \overbrace{\frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}}^{H.O.} + \frac{E_C}{2}\Big(\overbrace{1 - 2(N_g + N_g^{qu})}^{=0 \text{ at } N_g = \frac{1}{2}}\Big)\hat{\sigma}_z - \frac{E_J}{2}\hat{\sigma}_x$$

N_g^{qu} : quantum fluctuations of charge on capacitor C_g

$N_g = \frac{1}{2}$: consider charge degeneracy

quantum fluctuations of harmonic oscillator

$$\hat{H}_{HO} = \frac{1}{2} C \hat{V}^2 + \frac{1}{2} L \hat{I}^2$$

- consider only voltage fluctuations with operator $\hat{V}$

$$\hat{V} = \sqrt{\frac{\hbar\omega_r}{2C}}\,(\hat{a}^+ + \hat{a})$$

- variance of voltage fluctuations

$$\Delta V^2 = \langle 0|\hat{V}^2|0\rangle - \underbrace{\langle 0|\hat{V}|0\rangle}_{\text{mean voltage } = 0 \text{ for } |n\rangle = |0\rangle}{}^2$$

$$= \frac{\hbar\omega_r}{2C}\underbrace{\langle 0|(\hat{a}^+ + \hat{a})^2|0\rangle}_{=0} = \frac{\hbar\omega_r}{2C}$$

- with quantum fluctuations of charge

$$N_g^{qu} = \frac{C_g}{2e}\hat{V}^{qu} = \frac{C_g}{2e}\sqrt{\frac{\hbar\omega_r}{2C}}\,(\hat{a}^+ + \hat{a})$$

Full Hamiltonian

- with change of basis $\hat{\sigma}_z \rightarrow \hat{\sigma}_x$ and $\hat{\sigma}_x \rightarrow -\hat{\sigma}_z$

$$\hat{H} = \hbar\omega_r\left(\hat{a}^+ a + \frac{1}{2}\right) + \frac{E_C}{2}\frac{C_g}{2e}\sqrt{\frac{\hbar\omega_r}{2C}}\left(\hat{a}^+ + \hat{a}\right)\hat{\sigma}_x + \frac{E_J}{2}\hat{\sigma}_z$$

- and qubit raising and lowering operators $\hat{\sigma}^+, \hat{\sigma}^-$

$\hat{\sigma}_x = (\hat{\sigma}^+ + \hat{\sigma}^-)$ we find for the interaction

$$\frac{E_c}{2}\frac{C_g}{e}\sqrt{\frac{\hbar\omega_r}{2C}}\left(\cancelto{0}{\hat{a}^+\hat{\sigma}^+} + \hat{a}^+\hat{\sigma}^- + \hat{a}\hat{\sigma}^+ + \cancelto{0}{\hat{a}\hat{\sigma}^-}\right)$$

rotating wave approximation (RWA)

- with $E_c = \dfrac{(2e)^2}{2C_\Sigma}$ the full Hamiltonian reads

$$\hat{H} = \hbar\omega_r\left(\hat{a}^+\hat{a}\right) + \hbar g\left(\hat{a}^+\hat{\sigma}^- + \hat{a}\,\hat{\sigma}^+\right) + \frac{E_J}{2}\hat{\sigma}_z$$

with coupling constant

$$\hbar g = \frac{C_g}{C_\Sigma}\,2e\sqrt{\frac{\hbar\omega_r}{2C}}$$

where

$\dfrac{2g}{2\pi}$ is the vacuum Rabi frequency

The dispersive approximation of the Jaynes-Cummings Hamiltonian

$$H_{JC} = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\omega_q}{2}\sigma_z + \hbar g\left(a^\dagger\sigma^- + \sigma^+ a\right)$$

Calculate

$$\tilde{H} = U H U^\dagger$$

to 2nd order in g

with $U = \exp \frac{g}{\Delta}\left[a\sigma^+ - a^\dagger\sigma^-\right]$ and $\Delta = \omega_q - \omega_r$

results in the dispersive JC-Hamiltonian

$$\tilde{H} \approx \hbar\Big[\omega_r + \underbrace{\frac{g^2}{\Delta}\sigma_z}_{\text{dispersive shift}}\Big] a^\dagger a + \frac{\hbar}{2}\Big[\omega_q + \underbrace{\frac{g^2}{\Delta}}_{\text{Lamb shift}}\Big]\sigma_z$$

see A. Blais et al. Physical Review A **69**, 062320 (2004)

Driving transitions of a qubit in a cavity

$$H_{\mu w} = \hbar \underset{\Uparrow}{\varepsilon(t)} \left(a^{+} e^{-i\omega_d t} + a\, e^{i\omega_d t} \right)$$

amplitude of
the cavity drive field

$H_{JC} + H_D$ is the Hamiltonian of the driven system

using the unitary transform for the dispersive approximation one finds

$$U \left(H_{JC} + H_D \right) U^{+}$$

$$= \frac{\hbar}{2} \Big[\underbrace{\omega_q + 2\frac{g^2}{\Delta}\left(a^{+}a + \frac{1}{2}\right)}_{\text{qubit resonance frequency } \tilde{\omega}_q} - \omega_d \Big] \sigma_z + \hbar \underbrace{\frac{g\,\varepsilon(t)}{\Delta}}_{\text{qubit Rabi frequency}} \sigma_x$$

$$+ \hbar(\omega_r - \omega_d)\, a^{+}a + \hbar\, \varepsilon(t)\, (a^{+} + a)$$

for $\tilde{\omega}_q = \omega_d$ Rabi oscillations are induced in the qubit

see A. Blais et al. PRA **69**, 062320 (2004)