Physics IV – Mock Exam

SS 2007 – SOLUTIONS.

04/07/2007

①

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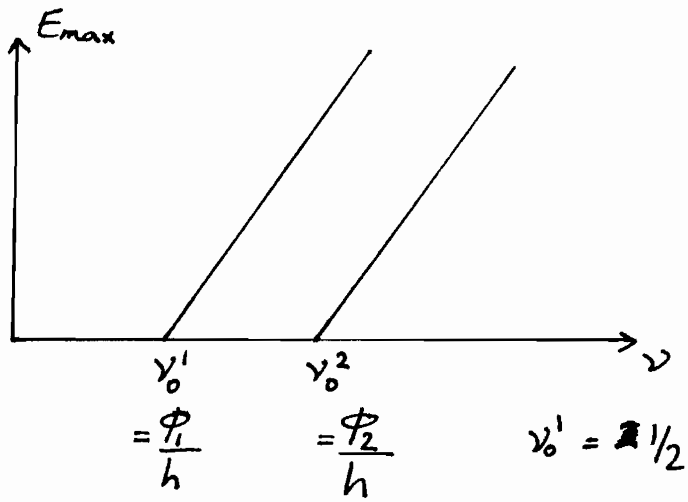


$\nu_0^1 = 1/2\, \nu_0^2$

$$E_{max} = h(\nu - \nu_0)$$

main features:

- the two lines have the same slope h
- the intercept with the x-axis gives work function $\varphi = \nu_{intercept} \times h$
- if incident photons do not have energy $> \varphi$, they cannot excite electrons out of the metal $\Rightarrow$ zero current in photoelectric experiment & $E_{max} = 0$
- maximum energy that the electrons can have is $h\nu - \varphi$ since one photon is absorbed each electron.

ⓑ $\varphi_{tungsten} = 5.4\,eV$ $\quad \lambda = 175\,nm$ $\quad E_{max} = 1.7\,eV$

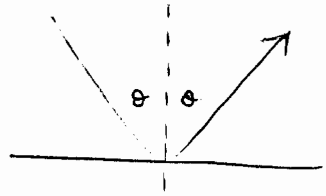
$$E_{max} = h\nu - \varphi \quad \Rightarrow \quad h = \frac{E_{max} + \varphi}{\nu}$$

$$\nu = c/\lambda \quad \Rightarrow \quad h = \frac{(E_{max} + \varphi)}{c} \times \lambda = (1.7 + 5.4) \frac{\times 175 \times 10^{-9}}{3 \times 10^{8}}\ eV.s$$

$$= 4.14 \times 10^{-15}\ eV.s$$

$$= 6.63 \times 10^{-34}\ J.s$$

② $E_{electron} = 50\text{ eV}$ $a = 0{\cdot}22\text{ nm}$



Bragg reflection $2a\cos\theta = n\lambda$

[NB: $\cos\theta$ when θ is measured from the normal.]

Need to know $\lambda_{electron}$:–

de Broglie $p = h/\lambda$ and $E = p^2/2m$

$$\Rightarrow E = \frac{h^2}{2m\lambda^2} \Rightarrow \lambda_{electron} = \sqrt{\frac{h^2}{2mE_{electron}}}$$

$$\theta = \cos^{-1}\left(\frac{\lambda}{2a}\right) = \cos^{-1}\left[\frac{h}{2a}\frac{1}{\sqrt{2mE_{electron}}}\right]$$

(for first maximum $n=1$) $= 67^\circ$

ⓑ X-rays must have the same wavelength as the electrons.

$$\lambda = \frac{h}{\sqrt{2mE_{electron}}} = 1{\cdot}74 \times 10^{-10}\text{ m}$$

energy of X rays (photons) $E = h\nu = \frac{hc}{\lambda} = \frac{6{\cdot}63\times10^{-34}\times3\times10^{8}}{1{\cdot}74\times10^{-10}}$

$= 1{\cdot}145 \times 10^{-15}\text{ J}$

$= 7{\cdot}2\text{ keV}$

③ The combined wavefunction of two fermions must be antisymmetric with respect to changing the positions of the two fermions.

ie Swapping the positions of the two fermions inverts the sign of the wavefunction.

Proof that two fermions cannot occupy the same state :–

– representing single particle state i by ψ_i

– labelling different fermions as A, B.. $\Rightarrow$ ψ_i^A means fermion A in state i.

Total wavefunction must be antisymmetric :–

$$\psi_{total} = \frac{1}{\sqrt{2}}\left(\psi_i^A \psi_j^B - \psi_i^B \psi_j^A\right)$$

When $i = j$,

$$\psi_{total} = \frac{1}{\sqrt{2}}\left(\psi_i^A \psi_i^B - \psi_i^B \psi_i^A\right) = 0$$

$\Rightarrow$ two fermions cannot exist simultaneously in the same single particle state.

④ 3D box with ∞ hard walls.

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$$\psi = A \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)$$

$$\int_0^a\int_0^a\int_0^a \psi^* \psi \, dx\,dy\,dz = 1$$

$$\Rightarrow A^2 \int_0^a \sin^2\left(\frac{n_x \pi x}{a}\right) dx \int_0^a \sin^2\left(\frac{n_y \pi y}{a}\right) dy \int_0^a \sin^2\left(\frac{n_z \pi z}{a}\right) dz = 1$$

$$\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \int_0^a \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{a}\right) dx = \left[\frac{x}{2} - \frac{1}{2}\left(\frac{a}{2n\pi}\right)\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{a}{2} - \frac{1}{2}\left(\frac{a}{2n\pi}\right)\sin(2n\pi) = \frac{a}{2}$$

↳ n is an integer ⇒ $\sin(2n\pi) = 0$

$$\int_0^a\int_0^a\int_0^a \psi^* \psi \, dx\,dy\,dz = A^2\left(\frac{a}{2}\right)^3 = 1$$

$$\Rightarrow \boxed{A = \frac{2\sqrt{2}}{a^{3/2}}}$$

(4)

(b) time-indep. Schrödinger equation : $H\Psi = E\Psi$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = E\Psi$$

Inside the box $V=0$ $\Rightarrow$ $-\frac{\hbar^2}{2m}\nabla^2\Psi = E\Psi$

$$\nabla^2\Psi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi = \left(-\frac{n_x^2\pi^2}{a^2}-\frac{n_y^2\pi^2}{a^2}-\frac{n_z^2\pi^2}{a^2}\right)\Psi$$

$$= -\frac{\pi^2}{a^2}\left(n_x^2+n_y^2+n_z^2\right)\Psi$$

$$\Rightarrow E = \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2+n_y^2+n_z^2\right)$$

(c) possible values for n_x, n_y & n_z :

	n_x	n_y	n_z	E
ground state ←	1	1	1	$\frac{3\hbar^2\pi^2}{2ma^2}$
1st excited state	2	1	1	$\frac{3\hbar^2\pi^2}{ma^2}$
	1	2	1	$\frac{3\hbar^2\pi^2}{ma^2}$
	1	1	2	$\frac{3\hbar^2\pi^2}{ma^2}$
	2	2	1	$\frac{9\hbar^2\pi^2}{2ma^2}$
		⋮		⋮

3 different wavefunctions
×2 for spin ←

$\Rightarrow$ degeneracy of the 1st excited state is 6

(4)

(d) Particle is in the ground state : $n_x = n_y = n_z = 1$

$$\text{Probability } P = \int_0^{a/4}\int_0^{a/4}\int_0^{a/4} \psi^* \psi \, dx \, dy \, dz$$

$$\int_0^{a/4} \sin^2\left(\frac{\overset{1}{\cancel{n}}\pi x}{a}\right) dx = \int_0^{a/4} \frac{1}{2}\left(1 - \cos\left(\frac{2\pi x}{a}\right)\right) dx$$

$$= \left[\frac{x}{2} - \frac{1}{2}\left(\frac{a}{2\pi}\right)\sin\left(\frac{2\pi x}{a}\right)\right]_0^{a/4}$$

$$= a/8 - a/4\pi \, \sin(\pi/2) = a\left(\frac{1}{8} - \frac{1}{4\pi}\right)$$

$$\Rightarrow P = \underset{A^2}{\underset{\downarrow}{\frac{8}{a^3}}} \times \left(\frac{a}{8} - \frac{a}{4\pi}\right)^3 = \boxed{\left(\frac{1}{4} - \frac{1}{2\pi}\right)^3 = 0.00075}$$

Classically the probability of finding the particle in a volume V is the ration of V to the total volume, therefore

$$P_{classical} = \frac{(a/4)^3}{a^3} = \left(\frac{1}{4}\right)^3 = 0.015$$

$\Rightarrow$ the QM probability is much lower due to the Sin shape of the wave-function.

⑤ $i\hbar \frac{d}{dt}\psi = \mathcal{H}\psi$

(a) $\psi(t=0) = \psi_1$

$\psi(t) = a(t)\,\psi_1 + b(t)\,\psi_2$

$$i\hbar\left(a'(t)\,\psi_1 + b'(t)\,\psi_2\right) = a(t)\,E_1\,\psi_1 + b(t)\,E_2\,\psi_2$$

ψ_1 & ψ_2 orthogonal $\therefore$ separate variables :-

$$\begin{cases} i\hbar\, a'(t) = E_1\, a(t) \\ i\hbar\, b'(t) = E_2\, b(t) \end{cases} \qquad \begin{array}{l} a(0) = 1 \\ b(0) = 0 \end{array}$$

$$\int_{a(0)}^{a(t)} \frac{1}{a}\,da = \int_0^t -\frac{iE_1}{\hbar}\,dt$$ & similar for $b(t)$

$$\Rightarrow a(t) = a(0)\,e^{-iE_1t/\hbar} = e^{-iE_1t/\hbar}$$

$$b(t) = 0$$

$$\Rightarrow \psi(t) = e^{-iE_1t/\hbar}\,\psi_1 \quad ///$$

(b) A measured at $t=0$

$$\phi_1 = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$\phi_2 = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

$$\Rightarrow \psi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) = x\,\phi_1 + y\,\phi_2$$

$$\psi_2 = \frac{1}{\sqrt{2}}(\phi_2 - \phi_1)$$

Probabilities of measuring a_1, $P(a_1) = |x|^2 = \frac{1}{2}$ ///

" " " a_2, $P(a_2) = |y|^2 = \frac{1}{2}$ ///

(c) a_1 is the measurement result $\Rightarrow$

$$\psi(0) = \phi_1 = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$\psi(t) = \frac{1}{\sqrt{2}}\left(e^{-iE_1t/\hbar}\psi_1 + i e^{-iE_2t/\hbar}\psi_2\right)$$

$$= \frac{1}{2}\left(e^{-iE_1t/\hbar}(\phi_1+\phi_2) + (i)^2 e^{-iE_2t/\hbar}(\phi_2-\phi_1)\right)$$

$$= \frac{1}{2}\left((\phi_1+\phi_2) + (\phi_1-\phi_2)e^{-i(E_2-E_1)t/\hbar}\right) e^{-iE_1t/\hbar}$$

$\downarrow$ ignore "global" phase

$$\psi(t) = \frac{1}{2}\left((1+e^{-i\omega t})\phi_1 + (1-e^{-i\omega t})\phi_2\right)$$

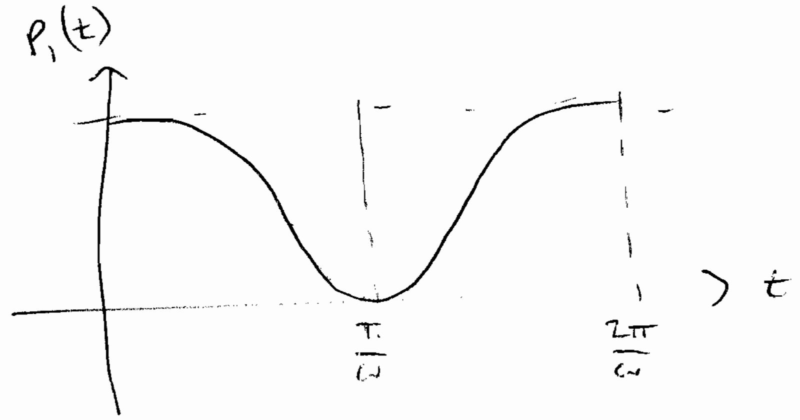
$$= x\phi_1 + y\phi_2 \qquad \text{where} \quad \omega = (E_2-E_1)/\hbar$$

$$P_1(t) = |x|^2$$

$$= \frac{1}{4}(1+e^{-i\omega t})(1+e^{i\omega t})$$

$$= \frac{1}{4}(1+1+e^{-i\omega t}+e^{i\omega t})$$

$$= \frac{1}{2}(1+\cos\omega t) \quad //$$



⑥ Zeeman Effect $\quad \Delta E = \dfrac{eB}{2m_e}\left(L_z + 2S_z\right)$

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$L_z = \hbar m_l$

$S_z = \hbar m_s$

$n=1$ energy levels : 1s only.

$n=2$ energy levels : 2s and 2p.

$$1s :- \quad m_l = 0, \; m_s = \pm \tfrac{1}{2} \quad \Rightarrow \quad \Delta E_{1s} = \pm \frac{eB\hbar}{2m_e}$$

$$2s :- \quad m_l = 0, \; m_s = \pm \tfrac{1}{2} \quad \Rightarrow \quad \Delta E_{2s} = \pm \frac{eB\hbar}{2m_e}$$

$$2p :- \quad m_l = 0, \pm 1, \; m_s = \pm \tfrac{1}{2}$$

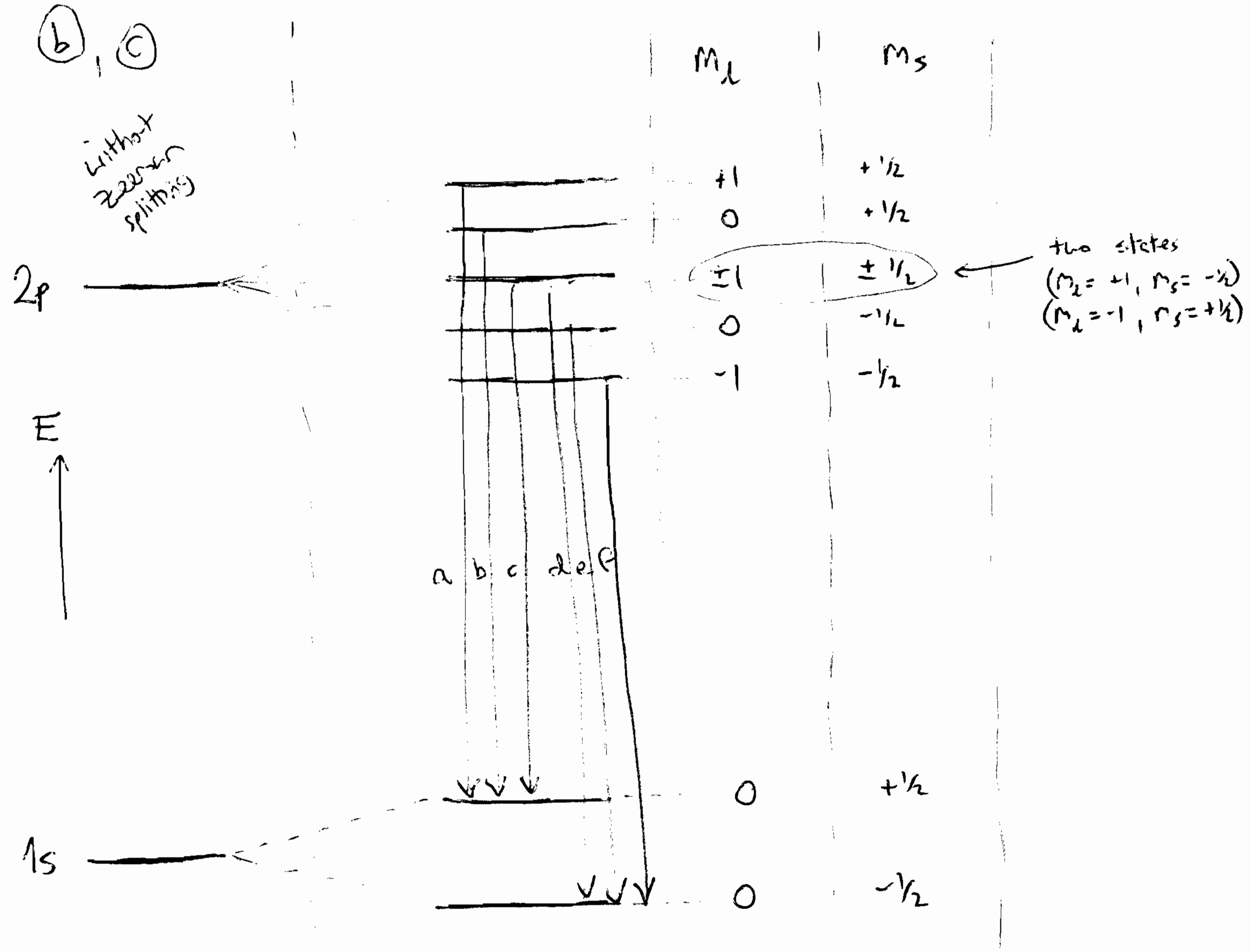
$$\Rightarrow \Delta E_{2p} = \frac{eB\hbar}{2m_e} \times \begin{array}{ccc} m_l & & m_s \\ \left\{\begin{matrix} -1 \\ -1 \\ 0 \\ 0 \\ +1 \\ +1 \end{matrix}\right\} & +2 & \left\{\begin{matrix} +\frac{1}{2} \\ -\frac{1}{2} \\ +\frac{1}{2} \\ -\frac{1}{2} \\ +\frac{1}{2} \\ -\frac{1}{2} \end{matrix}\right\} \end{array} = \frac{eB\hbar}{2m_e} \times \left\{\begin{matrix} 0 \\ -2 \\ +1 \\ -1 \\ +2 \\ 0 \end{matrix}\right\}$$

$$= (-2, -1, \underbrace{0}_{\text{degeneracy 2}}, +1, +2) \times \frac{eB\hbar}{2m_e}$$

(6 different states, but 5 different energies.)

pto for ⓑ, ⓒ and ⓓ

(b), (c)



Allowed transitions :- m_s conserved.

6 transitions shown above, labelled a–f.

Energy of transitions : $\Delta E_a = \Delta E_d$
$\Delta E_b = \Delta E_e$
$\Delta E_c = \Delta E_f$

$\Rightarrow$ 3 spectral lines only

(d) $B = 10\,T$

Splitting is $\Delta E = \dfrac{eB\hbar}{2m_e} = h\nu$

$$\nu = \frac{eB}{4\pi m_e} = 140\ \text{GHz}$$

(7) $$E_J = \frac{L^2}{2I} = \frac{J(J+1)\hbar^2}{2\mu r^2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$ where $m_1 = m_H$ (Hydrogen)
$m_2 = 19 m_H$ (Fluorine)

$$\Rightarrow \mu = \frac{19}{20} m_H$$

lowest frequency spectral line $\Rightarrow$ lowest energy transition
$\Rightarrow$ $J=0$ to $J=1$ transition

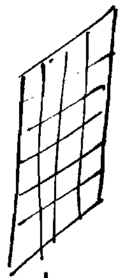
$$\Delta E = \frac{1(1+1)\hbar^2}{2 \cdot \frac{19}{20} m_H r^2}$$ & need to find $\boxed{r}$.

$$r^2 = \frac{\hbar^2}{\frac{19}{20} m_H \Delta E}$$ & $\Delta E = h\nu$
where $\nu = 1.25 \times 10^{12}$ Hz.

$$\Rightarrow r = \sqrt{\frac{\hbar^2}{\frac{19}{20} m_H h\nu}} = \sqrt{\frac{\hbar}{\frac{19}{20} m_H \cdot 2\pi \cdot \nu}}$$

$$= 9.2 \times 10^{-11}\ \text{m}$$

(8)



Solar panel
on Earth
area $A = 10\,m^2$
efficiency $= 30\%$

Earth to Sun distance $d_{E-S} = 150 \times 10^6$ km

Sun's temperature $T_S = 6000$ K

Sun's radius $R_S = 7 \times 10^5$ km

total power per unit area emitted from a black-body at temperature T is given by the Stefan-Boltzmann law:

$$\frac{P}{\text{unit area}} = \sigma T^4$$

treating the Sun as a black-body, the total power emitted by the Sun is: $P_{total} = \sigma T_S^4 \times 4\pi R_S^2$

$\Rightarrow$ power per unit area falling on Earth from the Sun is:

$$\frac{P_{\text{on Earth}}}{\text{unit area}} = \frac{P_{total}}{4\pi\, d_{E-S}^2} = \sigma T_S^4 \times \frac{R_S^2}{d_{E-S}^2}$$

the total power output of the Solar Panel is then given by:

$$\underbrace{0.3}_{\text{efficiency}} \times \underbrace{A}_{\text{area of the panel}} \times \sigma T_S^4 \times \frac{R_S^2}{d_{E-S}^2} = 0.3 \times 10 \times 5.67 \times 10^{-8} \times 6000^4 \times \left(\frac{7 \times 10^5}{150 \times 10^6}\right)^2$$

$$= \underline{\underline{4.8\ \text{KW}}}$$