The Josephson Junction as a Non-Linear Inductor

induction law $V = -L\dot{I}$

Josephson equations $I = I_0 \sin\delta$ $\boxed{dc}$

$V = \frac{\Phi_0}{2\pi}\dot{\delta}$ $\boxed{ac}$

with $\dot{I} = I_0 \cos\delta\,\dot{\delta}$

follows $\boxed{V = \frac{\Phi_0}{2\pi I_0}\frac{1}{\cos\delta}\dot{I}} = L_J \dot{I}$

Josephson inductance $L_J = L_{J0} \frac{1}{\cos\delta}$ → non-linearity

$L_{J0} = \frac{\Phi_0}{2\pi I_0}$ specific Josephson inductance

Note: Phase difference δ in Josephson junction can be regarded as normalized magnetic flux

$$\delta = 2\pi \frac{\Phi}{\Phi_0}$$

Josephson Inductance and Josephson Energy

- Josephson energy

$$E_J = \int V I \, dt$$

$$= \int \frac{\Phi_0}{2\pi} \dot{\delta} \, I_0 \sin\delta \, dt$$

$$= \frac{\Phi_0 I_0}{2\pi} \cos\delta$$

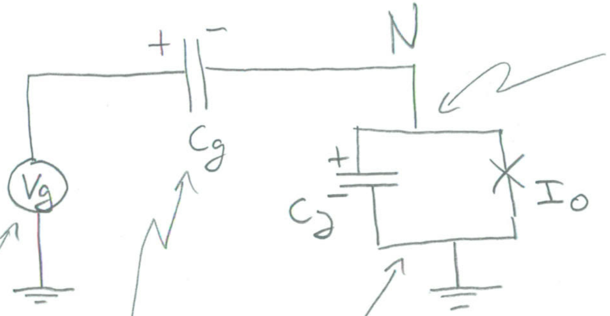
$$= E_{J0} \cos\delta \qquad \text{with } E_{J0} = \frac{\Phi_0 I_0}{2\pi}$$

- typical parameters: $I_0 = 100 \, mA$

$$\Rightarrow L_{J0} = \frac{\Phi_0}{2\pi I_0} \approx 3 \, mH \qquad (\sim 3 \text{ mm of wire})$$

$$\Rightarrow E_{J0} = \frac{\Phi_0 I_0}{2\pi} \approx 50 \, GHz$$

The Cooper Pair Box Qubit



island on which charges are localized

gate voltage

gate capacitor

grounded reservoir of charge carriers

discrete variable

$$\boxed{N = \frac{Q}{2e}}$$

number of Cooper pairs on island (with respect to charge neutrality)

$$\boxed{N_g = \frac{C_g V_g}{2e}}$$

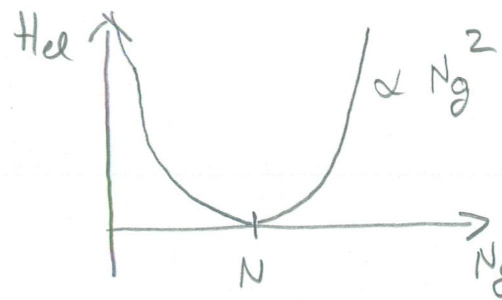
polarization charge on gate capacitor

continuous variable

- Hamiltonian $\quad H = \dfrac{Q^2}{2C} + \dfrac{\Phi^2}{2L} =$ electrostatic + magnetic energy

- electrostatic energy

$$H_{el} = \frac{Q^2}{2C} = \frac{(2e)^2 (N - N_g)^2}{2 C_\Sigma}$$



with $\quad C_\Sigma = C_J + C_g + \ldots$ (stray capacitances) $\quad$ total capacitance of island

and $\quad E_C = \dfrac{(2e)^2}{2 C_\Sigma} \quad$ charging energy

How does the electrostatic energy depend on the gate voltage?

- magnetic energy

$$H_{mag} = -E_J \cos\delta = -\frac{\Phi_0 I_0}{2\pi}\overbrace{\cos\delta}^{\approx 1-\frac{\delta^2}{2}+\dots}$$

$$\approx -\frac{\Phi_0 I_0}{2\pi}\left(1-\frac{1}{2}\left(\frac{\Phi}{\Phi_0}2\pi\right)^2+\dots\right)$$

$$\approx \frac{1}{2}\frac{\Phi^2}{L_{J0}}$$

(standard expression for mag. energy)

- Cooper pair box <u>Hamilton operator</u>

$$\hat{H} = E_C(\hat{N}-N_g)^2 - E_J\underbrace{\cos\hat{\delta}}_{\frac{1}{2}(e^{i\hat{\delta}}+e^{-i\hat{\delta}})}$$

with commutation relation $[\hat{\delta},\hat{N}] = i$ for conjugate variables δ and N.

<u>Transformation between bases:</u>

$$\boxed{|\delta\rangle = \frac{1}{\sqrt{2\pi}}\sum_N e^{iN\delta}|N\rangle}$$

number states

- properties of phase $\hat{\delta}$ and number $\hat{N}$ operators

$$[\hat{\delta},\hat{N}] = i \Rightarrow e^{\pm i\hat{\delta}}|N\rangle = |N\pm 1\rangle$$

number operator

$$\hat{N}|N\rangle = N|N\rangle$$

$$\underbrace{\sum_N |N\rangle\langle N| = \mathbb{1}}_{\text{completeness}} \quad ; \quad \underbrace{\langle M|N\rangle = \delta_{M,N}}_{\text{orthogonality}}$$

Hamilton Operator of Cooper Pair Box in Charge Basis

$$\hat{H} = \sum_N \Big(\underbrace{E_C (N - N_g)^2 |N\rangle\langle N|}_{\text{energy of charges on island}} - \underbrace{E_J/2 \left(|N\rangle\langle N+1| + |N+1\rangle\langle N| \right)}_{\text{energy to add or remove charges from island}} \Big)$$

Show band diag. slides!

- solve time independent Schrödinger equation in discrete charge basis $|N\rangle$ to find energy eigenstates $|\Psi\rangle$ of qubit

$$\hat{H} |\Psi_m\rangle = E_m |\Psi_m\rangle$$

- equivalent Hamilton operator in phase basis $\hat{\delta}$ (continuous & periodic)

$$\hat{N} = \frac{\hat{Q}}{2e} = -i\hbar \frac{1}{2e} \frac{\partial}{\partial \Phi} = -i \frac{\partial}{\partial \delta}$$

$$\hat{H} = E_C \left(-i \frac{\partial}{\partial \delta} - N_g \right)^2 - E_J \cos \delta$$

$\Rightarrow$ exact solutions for $H \Psi_m(\delta) = E_m \Psi_m(\delta)$ are Mathieu functions