The Hadamard Gate

generation of superposition from basis states

$$|0\rangle \;\text{———}\; \boxed{\hat{H}} \;\text{———}\; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \;\text{———}\; \boxed{\hat{H}} \;\text{———}\; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

- matrix representation

$$\hat{H} = \frac{1}{\sqrt{2}}\left(\hat{X} + \hat{Z}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note: - this gate is used in many quantum algorithms to prepare superposition states from basis states

Can you think of alternative ways to generate a superposition state?

Single Qubit Dynamics

spin 1/2 particle in external field

magnetic field

$\vec{B} = (0, 0, B_z)$

— magnetic moment

- Hamiltonian $\quad H = -\vec{\mu} \cdot \vec{B}$

- corresponding Operator $\quad \hat{H} = -\frac{g \mu_B B_z}{2} \hat{Z}$

g : gyromagnetic ratio

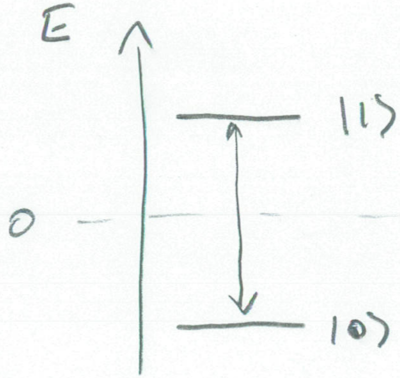
μ_B : Bohr magneton

- time independent Schrödinger equation $\quad \hat{H} |\psi_i\rangle = E_i |\psi_i\rangle$

- eigenstates of $\hat{H}$ are $|0\rangle$ and $|1\rangle$

$$\hat{H} |0\rangle = E_0 |0\rangle$$
$$\hat{H} |1\rangle = E_1 |1\rangle$$

- energy level diagram



$$E_1 = \frac{g \mu_B B_z}{2}$$

$$E_0 = -\frac{g \mu_B B_z}{2}$$

$$\Delta E = g \mu_B B_z = \hbar \Omega_z = E_1 - E_0$$

- time dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$$

How would you determine the dynamics of a system described by the operator $\hat{H}$?

- general solution for time independent $\hat{H}$

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

with
$$\exp(i\Theta \hat{O}) = \cos\Theta \, \hat{I} + i \sin\Theta \, \hat{O}$$

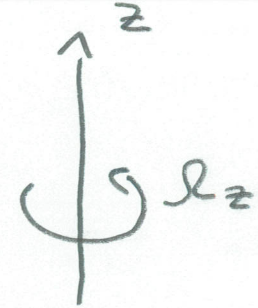
for Operators with $\hat{O}^2 = \hat{I}$ and $\Theta \in \mathbb{R}$

e.g. for all Pauli matrices

- for spin ½ example

$$\hat{H} = -\frac{\hbar \Omega_z}{2} \hat{Z}$$

rotation about z-axis on Bloch sphere!



$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

$$= \left(\cos\frac{\Theta_z}{2} \hat{I} + i \sin\frac{\Theta_z}{2} \hat{Z}\right) |\psi(0)\rangle = R_z(\Theta_z) |\psi(0)\rangle$$

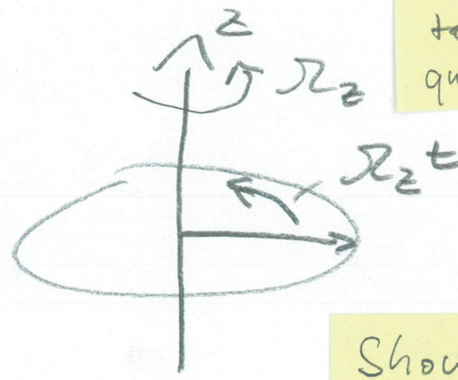
with $\Theta_z = \Omega_z t$

Dynamics of Superposition State

- initial state : $$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$
- Hamilton operator $$\hat{H} = -\frac{\hbar\Omega_z}{2}\underbrace{\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)}_{\hat{Z}}$$
- final state $$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi(0)\rangle$$
$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\Omega_z t}{2}}|0\rangle + e^{-i\frac{\Omega_z t}{2}}|1\rangle\right)$$
upto global phase $$= \frac{1}{\sqrt{2}}\left(|0\rangle + \underbrace{e^{-i\Omega_z t}}_{e^{i\varphi}}|1\rangle\right)$$

- on Bloch sphere

$$\Theta = \frac{\pi}{2}$$
$$\varphi = -\Omega_z t$$



"Do the Bloch sphere dance to illustrate qubit dynamics!"

Can you work out what dynamics the Hamiltonian $\hat{H}_x = -\frac{\hbar\Omega_x}{2}\hat{X}$ induces?

How is this useful for controlling the qubit state?

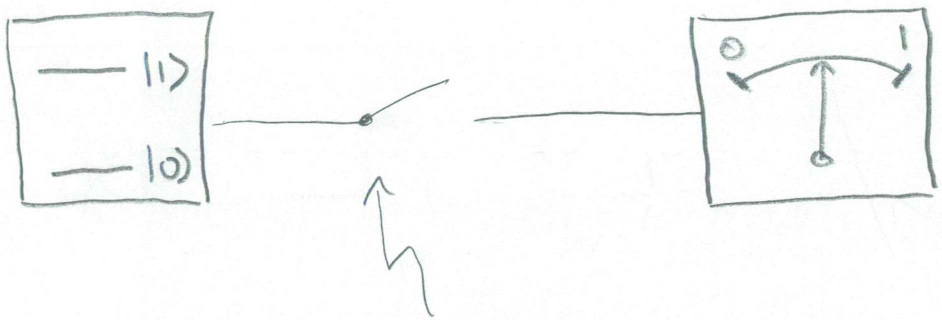
Show slides with other rotation operators.

Quantum Measurement

- generic set up

closed quantum system (QS)

measurement apparatus (MA)



controlled interaction

- goal: faithful reconstruction of qubit state

What properties do you suggest should an ideal measurement apparatus for a quantum bit have?

desired properties of measurement:

- ON / OFF: no interaction of MA with QS when OFF, strong interaction when ON
- high fidelity of mapping of QS state to MA state

$$|0\rangle \longrightarrow 0$$
$$|1\rangle \longrightarrow 1$$

- fast MA in comparison to coherence
- quantum non-demolition (QND): repeatability of measurement with same outcome

Measurement Postulate

- measurement result m with qubit in state $|\psi\rangle$ occurs with probability

$$P_m = \langle\psi| \hat{M}_m^\dagger \hat{M}_m |\psi\rangle$$

with a set of measurement operators $\{\hat{M}_m\}$ acting on the qubit states $|\psi\rangle$ that is complete

$$\sum_m P_m = 1 \iff \sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}$$

- post measurement qubit state

$$|\psi'\rangle = \frac{\hat{M}_m |\psi\rangle}{\sqrt{P_m}}$$

Measurement of Qubit State in Computational Basis

- define measurement operators

$$\hat{M}_0 = |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$\hat{M}_1 = |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}(0\ 1) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

complete $\sum_m \hat{M}_m^{+} M_m = \hat{I}$

What do you think one can learn from a single measurement on a single qubit? What would you propose to do to learn more about the qubit state?

- example: measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P_0 = \langle\psi|\hat{M}_0^{+}\hat{M}_0|\psi\rangle = \alpha^*\alpha = |\alpha|^2$$

$$P_1 = \langle\psi|\hat{M}_1^{+}\hat{M}_1|\psi\rangle = \beta^*\beta = |\beta|^2$$

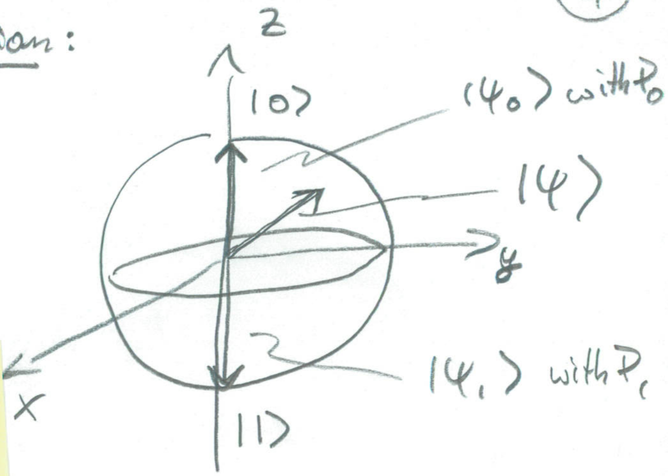
NOTE:
- single preparation of state $|\psi\rangle$ with single measurement $\hat{M}_m$ results in single outcome m with probability P_m
- to determine P_m, $|\psi\rangle$ has to be prepared and measured repeatedly (here determines $|\alpha|^2$ and $|\beta|^2$)
- full knowledge of state requires α, β to be known

- post measurement state

$$|\psi_0\rangle = \frac{\hat{M}_0|\psi\rangle}{\sqrt{p_0}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$|\psi_1\rangle = \frac{\hat{M}_1|\psi\rangle}{\sqrt{p_1}} = \frac{\beta}{|\beta|}|1\rangle$$

<u>interpretation:</u>



In your opinion does this type of measurement suffice to fully describe a qubit state?

- repeated measurement

$$P_{00} = \langle\psi_0|\hat{M}_0^\dagger\hat{M}_0|\psi_0\rangle = 1$$

probability of result of second measurement to be $m=0$ provided that first result was $m=0$

$$P_{01} = 0$$
$$P_{10} = 0$$
$$P_{11} = 1$$

What do you think could be reasons that measurement is not repeatable with same result?

<u>NOTE:</u> – any projective measurement should fulfill the above properties of QS

<u>PROBLEMS:</u>
- spontaneous emission or absorbtion in QS due to MA
- stimulated emission
- misidentification of state by measurement apparatus

Multiple Qubit States and Entanglement

How many different states can two classical or two quantum mechanical bits be in?

register of $n=2$ classical bits:

BIT A	BIT B
0	0
0	1
1	0
1	1

2^n different states

note: - only one state is realized at any given time

register of $n=2$ quantum bits

QUBIT $\|A\rangle$	QUBIT $\|B\rangle$
$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$

2^n basis states

BUT: - quantum register can be in any superposition of basis states

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formal description of general state of $n=2$ quantum register

$$|\psi\rangle = |A\rangle \otimes |B\rangle = |AB\rangle \text{ (according to } 4^{th} \text{ postulate)}$$

e.g. $|A\rangle = \alpha_A |0\rangle + \beta_A |1\rangle$; $|B\rangle = \alpha_B |0\rangle + \beta_B |1\rangle$

$$|\psi\rangle = \alpha_A \alpha_B |00\rangle + \alpha_A \beta_B |01\rangle + \beta_A \alpha_B |10\rangle + \beta_A \beta_B |11\rangle$$

with $\sum_{ij} |\alpha_{ij}|^2 = 1$ (normalization condition for probabilities)

Information Content of Many Qubit States

register of n qubits:

- 2^n basis states
- general superposition state is described by 2^n complex coefficients

How would you best describe the state of $n=500$ qubits? Is it at all possible?

Consider $n = 500$ qubits

- need $2^{500} = 3 \times 10^{150}$ coefficients

  ⇒ larger than number of atoms in universe

  ⇒ impossible to store information about state classically

This is why it is difficult to simulate QM on a classical computer. But it would be natural to simulate QM on a quantum computer.