

Entangled Qubit States

Definition: An entangled state of a composite system is a state that cannot be written as a product state of the component systems.

Product state: (example)

$$|\psi\rangle = |\psi_1, \psi_2\rangle$$

with $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$
 $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

Entangled state: (example)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

How would you figure out, if this is an entangled state?

Does nature create such states?
How would you go about creating such a state?

$$\Rightarrow \alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \wedge \beta_1\beta_2 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \alpha_1\beta_2 \neq 0 \wedge \alpha_2\beta_1 \neq 0$$

} i.e. not a product state

- Questions:
- How are such states created?
 - What are their properties?

Correlations of Entangled States

Measurement of individual qubit states in an entangled pair

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

- measure ground state (0) of first qubit (1)

What would you think is the result of a measurement of the state of both qubits?

$$P_1(0) = \langle \psi | (M_0 \otimes I)^\dagger (M_0 \otimes I) | \psi \rangle = \frac{1}{2}$$

qubit ↑ state

tensor products of individual qubit measured operators

- post measurement state

$$|\psi'\rangle = \frac{(M_0 \otimes I) |\psi\rangle}{\sqrt{P_1(0)}} = |00\rangle$$

- measure ground state (0) of second qubit (2) given that first one was measured in state (0).

$$P_2(0) = \langle \psi' | (I \otimes M_0)^\dagger (I \otimes M_0) | \psi' \rangle = 1$$

⇒ The outcomes of the measurements of both qubit states are 100% correlated. Such correlations are impossible in a classical system (compare with Bell inequalities)

Entanglement as a New Resource

Transmit two bits of classical information by sending one qubit between two parties Alice and Bob: Super Dense Coding

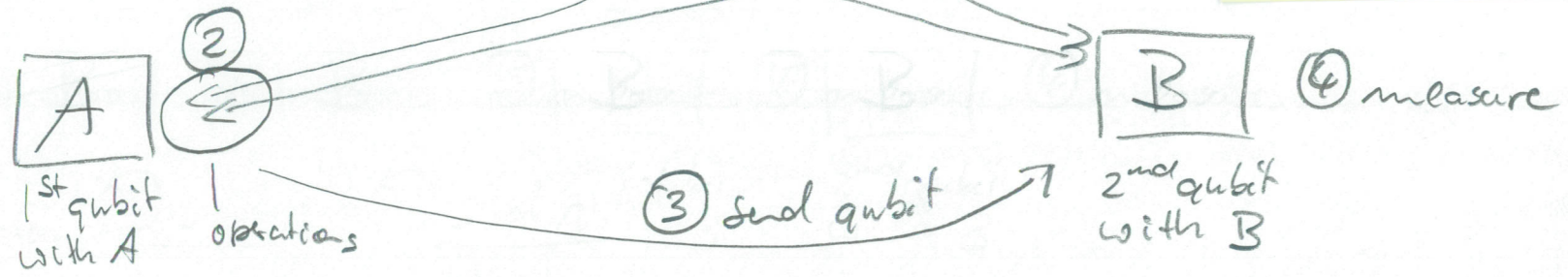
Protocol:

① Share entangled pair of qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

How is this better than classical?

original proposal by Wiesner and Bennett!



② Alice performs one of 4 local operations on her bit

$$\left. \begin{matrix} I_1 \otimes I_2 \\ Z_1 \otimes I_2 \\ X_1 \otimes I_2 \\ iY_1 \otimes I_2 \end{matrix} \right\} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \begin{cases} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \phi^+ & \longrightarrow 00 \\ \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = \phi^- & \longrightarrow 01 \\ \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle) = \psi^+ & \longrightarrow 10 \\ \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) = \psi^- & \longrightarrow 11 \end{cases}$$

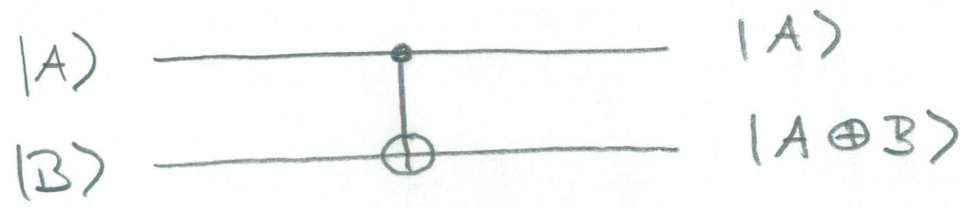
What about physical realization?
27 slides: realized with photons!

③ Alice sends qubit to Bob

④ Bob performs a measurement on both qubits and finds 4 outcomes

CNOT : A Universal 2-Qubit Logic Gate

Controlled NOT gate



INPUT

OUTPUT

truth table

|00>

|00>

|01>

|01>

|10>

|11>

|11>

|10>

|A, B>

|A, A ⊕ B>

general

addition mod 2

CONTROL QUBIT

TARGET QUBIT

How would you realize a CNOT operation between two qubits?

- is reversible (unitary)
- is universal
- can be realized using any two qubit interaction combined with single qubit manipulations

What is required on a physical level to realize conditional logic?
Do you know any types of interactions between quantum particles?

Universality

Any multi qubit logic gate can be composed of CNOT and single qubit gates (X, Y, Z, I).

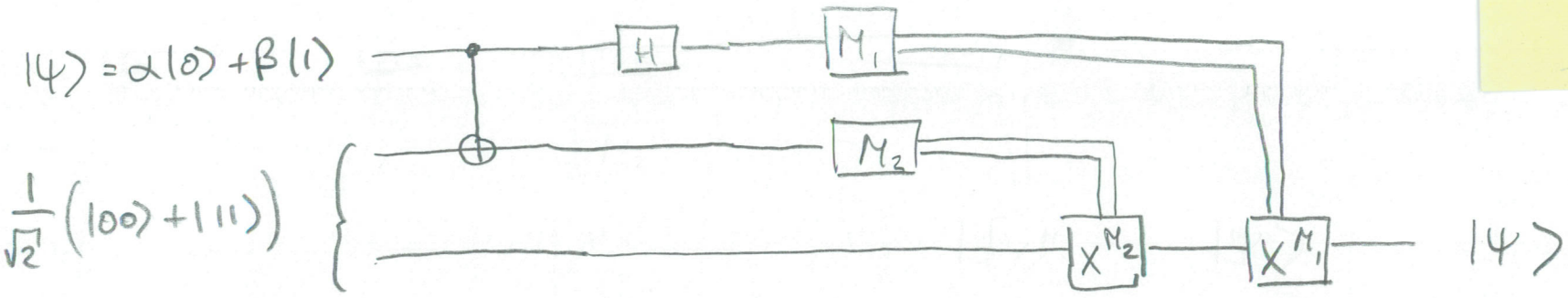
Quantum Teleportation

Task: Transfer an known quantum state $|\psi\rangle$ from Alice to Bob

Resources: entangled pair of qubits & classical communication

How would you perform this task?

Circuit:



INPUT (ALICE)

OUTPUT (BOB)

- Steps:
- (1) input
 - (2) CNOT
 - (3) Hadamard
 - (4) measurement
 - (5) conditional operations
 - (6) output

- Note:
- A has no information about $|\psi\rangle$ (and cannot obtain it)
 - state is always fully transferred

Teleportation Protocol

① Initial state $(\alpha|0\rangle + \beta|1\rangle) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (\alpha|1000\rangle + \alpha|1011\rangle + \beta|1100\rangle + \beta|1111\rangle)$

② $\xrightarrow{\text{CNOT}_{1,2}}$ $\frac{1}{\sqrt{2}} (\alpha|1000\rangle + \alpha|1011\rangle + \beta|1110\rangle + \beta|1101\rangle)$

③ $\xrightarrow{H_1}$ $\frac{1}{2} (\alpha|1000\rangle + \alpha|1100\rangle + \alpha|1011\rangle + \alpha|1111\rangle + \beta|1010\rangle - \beta|1110\rangle + \beta|1001\rangle - \beta|1101\rangle)$

What are the different measurement outcomes? With which prob. do they occur?

$$= \frac{1}{2} \left(\begin{aligned} &|00\rangle (\alpha|0\rangle + \beta|1\rangle) \\ &+ |10\rangle (\alpha|0\rangle - \beta|1\rangle) \\ &+ |01\rangle (\alpha|1\rangle + \beta|0\rangle) \\ &+ |11\rangle (\alpha|1\rangle - \beta|0\rangle) \end{aligned} \right)$$

④ measurement of qubit state
 $M_1 \otimes M_2 \otimes I$
 $P_{00} = P_{10} = P_{01} = P_{11} = \frac{1}{4}$

⑤ conditional qubit manipulations on post measurement state $|\psi'\rangle$

$$\left. \begin{aligned} |00\rangle &: \hat{I} |\psi'\rangle \\ |10\rangle &: \hat{Z} |\psi'\rangle \\ |01\rangle &: \hat{X} |\psi'\rangle \\ |11\rangle &: \hat{X} \hat{Z} |\psi'\rangle \end{aligned} \right\} = \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$

• requires transfer of two bits of classical information to Bob to perform local operations that recover the original state

Note:

- state of one qubit transferred using one pair of entangled qubits and two bits of classical information
- ↳ task cannot be performed classically

Applications:

- quantum error correction
- quantum gates
- quantum repeaters

original proposal : C.H. Bennett et al. Phys. Rev. Lett 70, 1895 (1993)

first experimental implementation

: D. Bouwmeester et al. Nature 390, 575 (1997)

↳ tested in different implementations using

- photons
- nuclear spins
- ions

↳ hallmark quantum information processing experiment