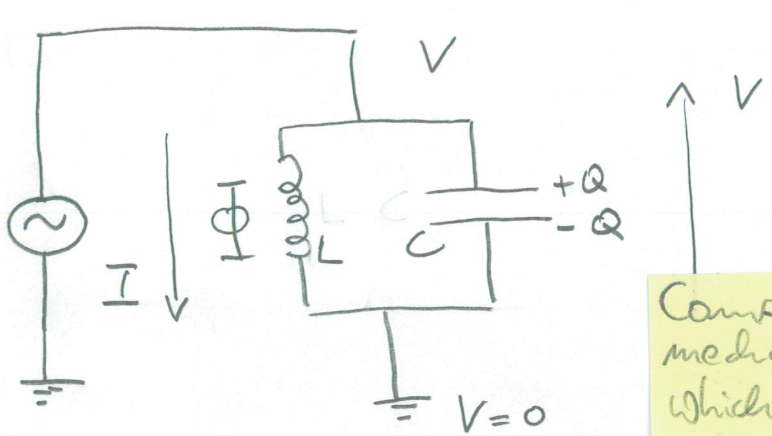
Quantum Information Processing with Electronic Circuits

goal: learn how to construct and operate electronic circuits that behave according to the laws of quantum mechanics

discuss:
- basic circuit elements and basic circuits
  - ↳ harmonic oscillator
  - ⇒ store photons on a chip
  - ⇒ discuss q.m. description of LC oscillator
- role of dissipation
- role of temperature

Electronic Harmonic Oscillator



Compare to mechanical oscillator. Which are the corresponding quantities?

- charge on capacitor

$$Q = CV$$

- flux in inductor

$$\Phi = LI$$

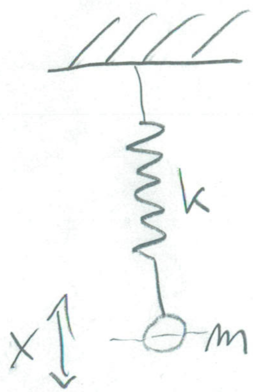
- Voltage across oscillator

$$V = \frac{Q}{C} = -L\dot{I} = -\dot{\Phi}$$

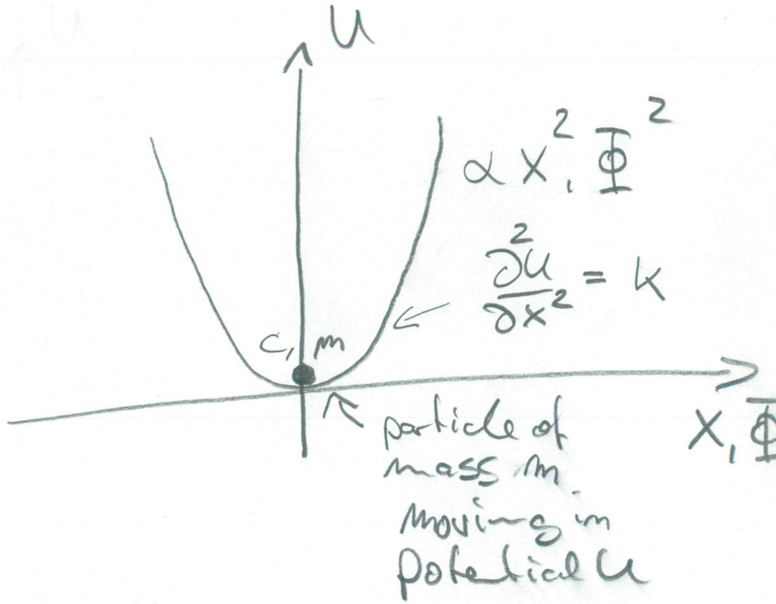
- Hamiltonian

$$H = \underbrace{\frac{CV^2}{2}}_{\text{electrostatic energy}} + \underbrace{\frac{LI^2}{2}}_{\text{magnetic energy}} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

- compare to mechanical harmonic oscillator



$$H = \underbrace{\frac{p^2}{2m}}_{\text{kinetic energy}} + \underbrace{\frac{kx^2}{2}}_{\text{potential energy}}$$



Characteristic Quantities

mechanical	electronic	harmonic oscillator
position x	flux Φ	conjugate variables
momentum p	charge Q	
mass m	capacitance C	
spring konstant k	inverse inductance $\frac{1}{L}$	
resonance frequency $\omega = \sqrt{\frac{k}{m}}$	$\omega = \frac{1}{\sqrt{LC}}$	

$$\frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I = \dot{Q}$$

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = V = -L\dot{I} = -\dot{\Phi}$$

- quantum mechanical operators:

$$\hat{x} = x \qquad \hat{\Phi} = \Phi$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \qquad \hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$$

We know how to quatize mechanical oscillator

- commutation relations

$$[\hat{x}, \hat{p}] = i\hbar \qquad [\hat{\Phi}, \hat{Q}] = i\hbar \qquad \text{flux - charge}$$

$$\Leftrightarrow \left[2\pi \frac{\hat{\Phi}}{\Phi_0}, \frac{\hat{Q}}{2e}\right] = [\hat{\delta}, \hat{N}] = i \qquad \text{phase - number}$$

Hamilton Operator

- using conjugate variables Q, Φ

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial \Phi^2} + \frac{1}{2L}\Phi^2$$

- using creation and annihilation operators

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q}^+ - i\hat{\Phi}^+\right) \qquad \text{creation operator}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q} + i\hat{\Phi}\right) \qquad \text{annihilation operator}$$

with $Z_c = \sqrt{L/C}$ impedance of oscillator

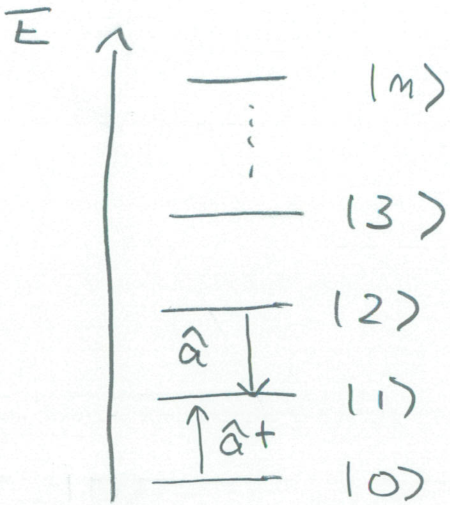
$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

Properties of $\hat{a}^+$ and $\hat{a}$

$$\hat{a}^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$\hat{a}\, |n\rangle = \sqrt{n}\, |n-1\rangle$$

$$\hat{a}^+ \hat{a}\, |n\rangle = n\, |n\rangle$$



Spectrum

with $|n\rangle$ number (Fock) state of harmonic oscillator

- relation to $\hat{Q}$ and $\hat{\Phi}$

$$\hat{Q} = \sqrt{\frac{\hbar}{2Z_c}}\left(\hat{a}^+ + \hat{a}\right)$$

$\Rightarrow$ relates to electric field stored on capacitor

$$\hat{\Phi} = i\sqrt{\frac{\hbar Z_c}{2}}\left(\hat{a}^+ - \hat{a}\right)$$

$\Rightarrow$ relates to magnetic field stored in inductor

or

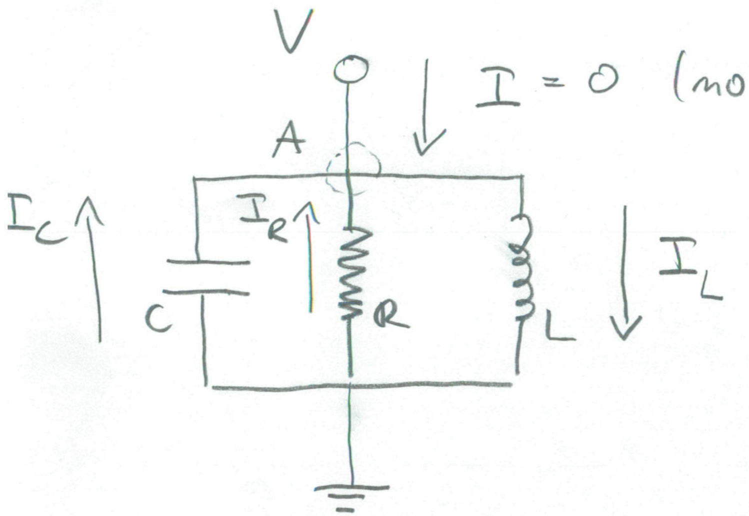
$$\hat{V} = \sqrt{\frac{\hbar\omega}{2C}}\left(\hat{a}^+ + \hat{a}\right)$$

$$\hat{I} = i\sqrt{\frac{\hbar\omega}{2L}}\left(\hat{a}^+ - \hat{a}\right)$$

with $\omega = \frac{1}{\sqrt{LC}}$ and $V = \frac{Q}{C}$ and $I = \frac{\Phi}{L}$

Dissipation in the Harmonic Oscillator

What is the role of dissipation in an electrical oscillator? How does it arise?



with

- current through resistor

$$I_R = V/R$$

- displacement current

$$I_C = \dot{Q}_C = C\dot{V}$$

- voltage across inductor

$$V = -L\dot{I}_L$$

- Kirchoff law at point A

$$I_L = I_R + I_C + I$$

$$\Leftrightarrow \quad -C\dot{V} - \frac{V}{R} + I_L = 0 \qquad \text{(same voltage at A)}$$

$$\Leftrightarrow \quad \boxed{\ddot{I}_L + \frac{1}{RC}\dot{I}_L + \frac{1}{LC} I_L = 0}$$

differential equation for current through inductor

- solutions

$$I_L(t) = I_L(0)\, e^{\lambda t} \qquad \text{with} \quad \lambda_{1,2} = \frac{1}{2LC}\left(-\frac{L}{R} \pm \sqrt{\left(\frac{L}{R}\right)^2 - 4LC}\right)$$

Energy Decay Rate

- underdamped oscillator $(4LC \gg L/R)$

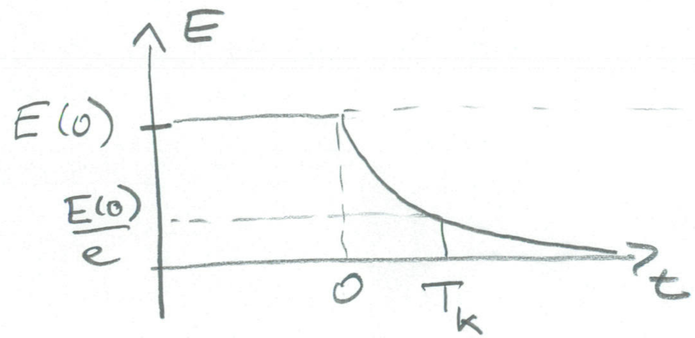
$$\lambda_{1,2} = -\frac{1}{2RC} \pm i \frac{1}{\sqrt{LC}} = -\alpha \pm i\omega$$

with $\alpha = \frac{1}{2RC} = \frac{1}{\tau}$ amplitude decay constant

$\tau = 2RC$ amplitude decay time

$\omega = 1/\sqrt{LC}$ oscillator frequency

- energy decay rate



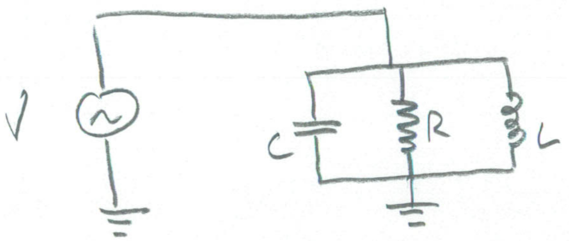
$$E \propto \frac{1}{2} L I_L^2 \propto e^{-\frac{1}{RC} t}$$

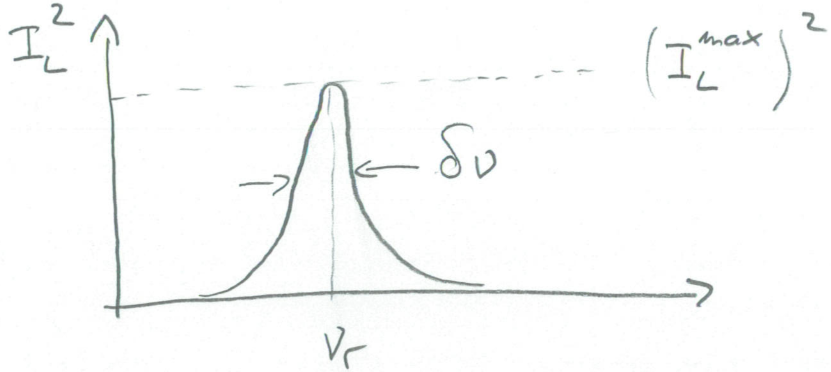
with $k = \frac{1}{RC}$ energy decay rate

$T_k = RC$ energy decay time

Spectral Response of Damped Harmonic Oscillator

- driven damped oscillator



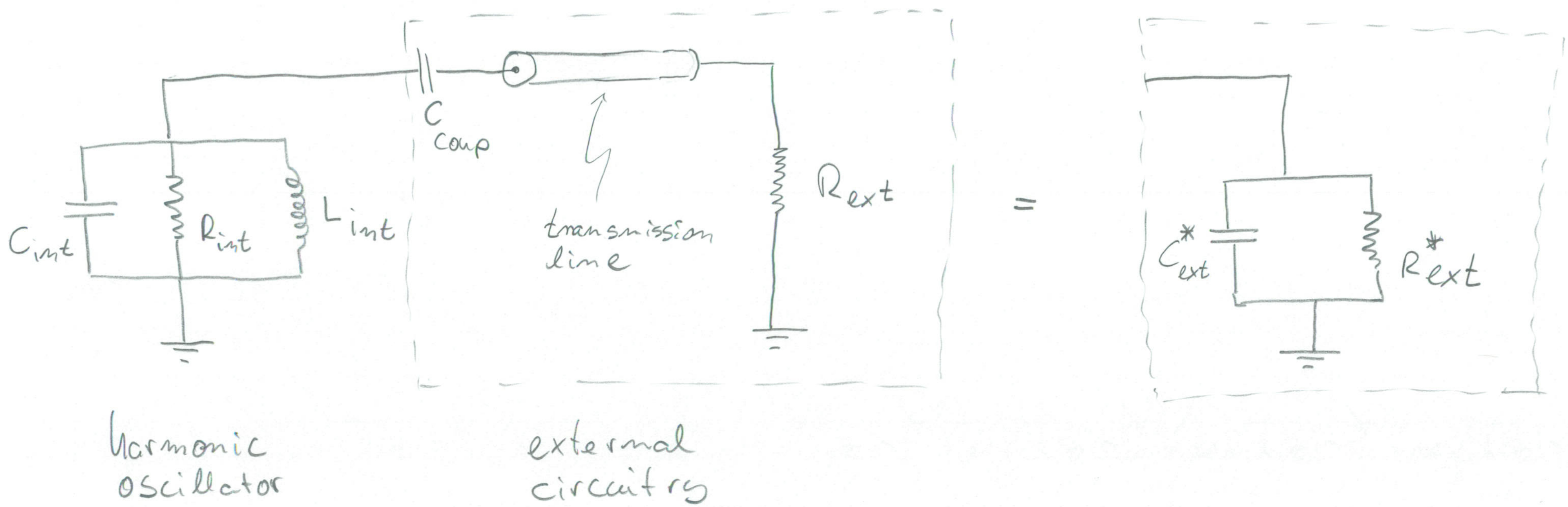


Lorentzian line shape

$$I_L^2(\nu) = (I_L^{max})^2 \frac{\delta\nu/\pi}{(\nu-\nu_r)^2 + \delta\nu^2}$$

with $\delta\nu$: full width of line at half maximum

Internal and External Dissipation



- total effective resistance $\frac{1}{R_{tot}} = \frac{1}{R_{int}} + \frac{1}{R^*_{ext}}$ ⇒ external contribution to energy decay
- total effective capacitance $C_{tot} = C_{int} + C^*_{ext}$ ⇒ frequency shift due to external circuit
- energy decay time of combined system $T_k = R_{tot} C_{tot}$

Show slides on Superconductivity and realizations of harmonic oscillators

The Josephson Junction as a Non-Linear Inductor

induction law $\quad V = -L\dot{I}$

Josephson equations $\quad I = I_0 \sin\delta \quad \boxed{dc}$

$$V = \frac{\Phi_0}{2\pi}\dot{\delta} \quad \boxed{ac}$$

with
$$\dot{I} = I_0 \cos\delta\,\dot{\delta}$$

follows
$$\boxed{V = \frac{\Phi_0}{2\pi I_0}\frac{1}{\cos\delta}\dot{I}} = L_J \dot{I}$$

Josephson inductance $\quad L_J = L_{J0}\frac{1}{\cos\delta} \longrightarrow$ non-linearity

$$L_{J0} = \frac{\Phi_0}{2\pi I_0} \quad \text{specific Josephson inductance}$$

Note: Phase difference δ in Josephson junction can be regarded as normalized magnetic flux

$$\delta = 2\pi\frac{\Phi}{\Phi_0}$$

Josephson Inductance and Josephson Energy

- Josephson energy

$$E_J = \int V I \, dt$$

$$= \int \frac{\Phi_0}{2\pi} \dot{\delta} \, I_0 \sin \delta \, dt$$

$$= \frac{\Phi_0 I_0}{2\pi} \cos \delta$$

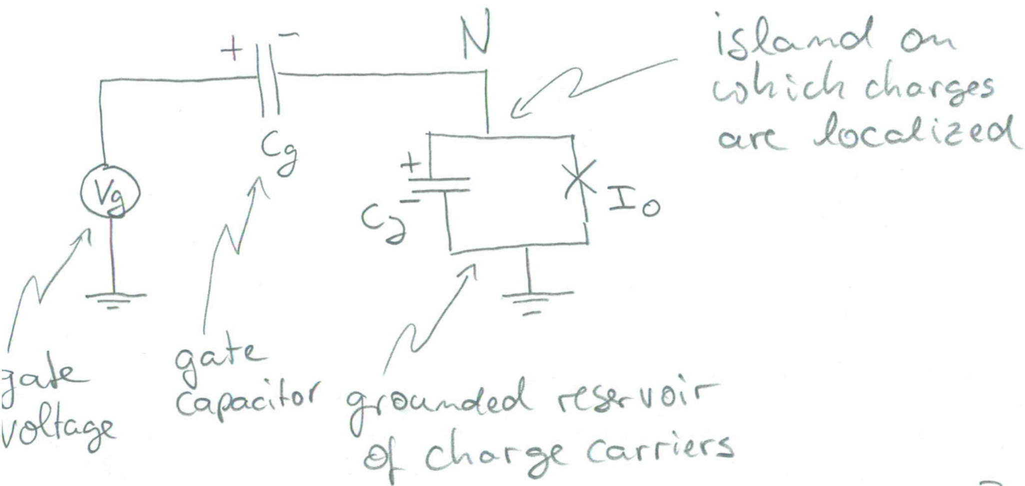
$$= E_{J0} \cos \delta \qquad \text{with } E_{J0} = \frac{\Phi_0 I_0}{2\pi}$$

- typical parameters: $I_0 = 100 \, \mu A$

$$\Rightarrow L_{J0} = \frac{\Phi_0}{2\pi I_0} \approx 3 \, \mu H \qquad (\sim 3 \text{ mm of wire})$$

$$\Rightarrow E_{J0} = \frac{\Phi_0 I_0}{2\pi} \approx 50 \text{ GHz}$$

The Cooper Pair Box Qubit



discrete variable

$$\boxed{N = \frac{Q}{2e}}$$

number of Cooper pairs on island (with respect to charge neutrality)

$$\boxed{N_g = \frac{C_g V_g}{2e}}$$

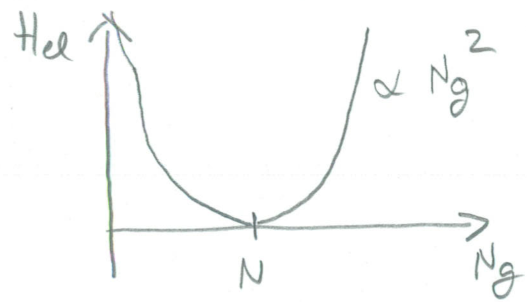
polarization charge on gate capacitor

continuous variable

- Hamiltonian $\quad H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} =$ electrostatic + magnetic energy

- electrostatic energy

$$H_{el} = \frac{Q^2}{2C} = \frac{(2e)^2 (N - N_g)^2}{2 C_\Sigma}$$



with $\quad C_\Sigma = C_J + C_g + \ldots$ (stray capacitances) $\quad$ total capacitance of island

and $\quad E_C = \frac{(2e)^2}{2 C_\Sigma} \quad$ charging energy

How does the electrostatic energy depend on the gate voltage?

• magnetic energy

$$H_{mag} = -E_J \cos\delta = -\frac{\Phi_0 I_0}{2\pi}\overbrace{\cos\delta}^{\approx 1-\frac{\delta^2}{2}+\dots}$$

$$\approx -\frac{\Phi_0 I_0}{2\pi}\left(1-\frac{1}{2}\left(\frac{\Phi}{\Phi_0}2\pi\right)^2+\dots\right)$$

$$\approx \frac{1}{2}\frac{\Phi^2}{L_{J0}}$$

(standard expression for mag. energy)

• Cooper pair box <u>Hamilton operator</u>

$$\hat{H} = E_C(\hat{N}-N_g)^2 - E_J \underbrace{\cos\hat{\delta}}_{\frac{1}{2}\left(e^{i\hat{\delta}}+e^{-i\hat{\delta}}\right)}$$

with commutation relation $[\hat{\delta},\hat{N}] = i$ for conjugate variables δ and N.

<u>Transformation between bases:</u>

$$\boxed{|\delta\rangle = \frac{1}{\sqrt{2\pi}}\sum_N e^{iN\delta}|N\rangle}$$

• properties of phase $\hat{\delta}$ and number $\hat{N}$ operators

number states

$$[\hat{\delta},\hat{N}] = i \Rightarrow e^{\pm i\hat{\delta}}|N\rangle = |N\pm 1\rangle$$

number operator

$$\hat{N}|N\rangle = N|N\rangle$$

$$\underbrace{\sum_N |N\rangle\langle N| = \mathbb{1}}_{\text{completeness}} \; ; \quad \underbrace{\langle M|N\rangle = \delta_{M,N}}_{\text{orthogonality}}$$

Hamilton Operator of Cooper Pair Box in Charge Basis

$$\hat{H} = \sum_N \Big(\underbrace{E_C (N - N_g)^2 \, |N\rangle\langle N|}_{\text{energy of charges on island}} - \underbrace{E_J/2 \big(|N\rangle\langle N+1| + |N+1\rangle\langle N| \big)}_{\text{energy to add or remove charges from island}} \Big)$$

- solve time independent Schrödinger equation in discrete charge basis $|N\rangle$ to find energy eigenstates $|\psi\rangle$ of qubit

Show band diag. Slides!

$$\hat{H} |\psi_m\rangle = E_m |\psi_m\rangle$$

- equivalent Hamilton operator in phase basis $\hat{\delta}$ (continuous & periodic)

$$\hat{N} = \frac{\hat{Q}}{2e} = -i\hbar \frac{1}{2e} \frac{\partial}{\partial \Phi} = -i \frac{\partial}{\partial \delta}$$

$$\hat{H} = E_C \left(-i \frac{\partial}{\partial \delta} - N_g \right)^2 - E_J \cos \delta$$

$\Rightarrow$ exact solutions for $H \psi_m(\delta) = E_m \psi_m(\delta)$ are Mathieu functions