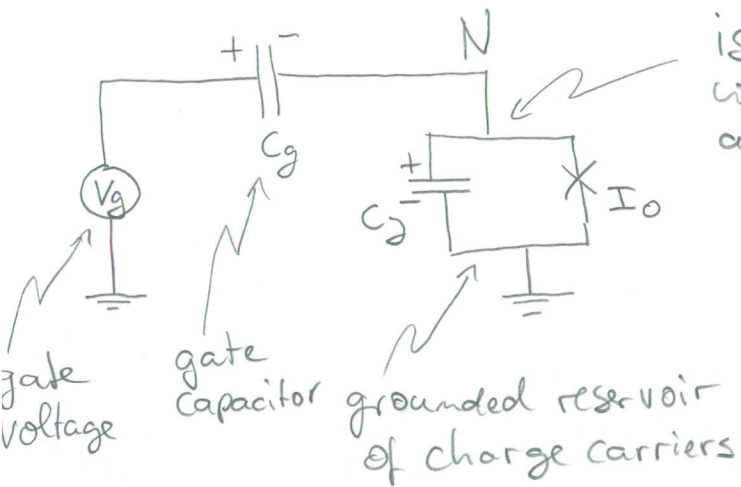


The Cooper Pair Box Qubit



island on which charges are localized

$$N = \frac{Q}{2e}$$

discrete variable

number of Cooper pairs on island (with respect to charge neutrality)

$$N_g = \frac{C_g V_g}{2e}$$

polarization charge on gate capacitor

continuous variable

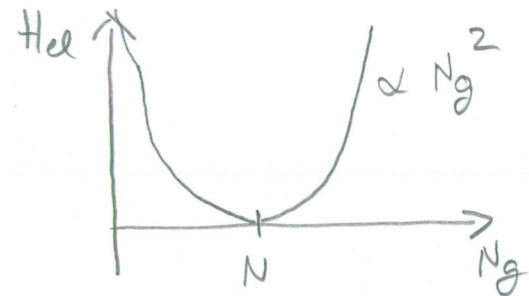
• Hamiltonian

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

electrostatic + magnetic energy

• electrostatic energy

$$H_{el} = \frac{Q^2}{2C} = \frac{(2e)^2 (N - N_g)^2}{2C_{\Sigma}}$$



with $C_{\Sigma} = C_j + C_g + \dots$ (stray capacitances)

total capacitance of island

and $E_C = \frac{(2e)^2}{2C_{\Sigma}}$ charging energy

How does the electrostatic energy depend on the gate voltage?

• magnetic energy

$$\begin{aligned}
H_{\text{mag}} &= -E_J \cos \delta = - \frac{\Phi_0 I_0}{2\pi} \overbrace{\cos \delta}^{\approx 1 - \frac{\delta^2}{2} + \dots} \\
&\approx - \frac{\Phi_0 I_0}{2\pi} \left(1 - \frac{1}{2} \left(\frac{\Phi}{\Phi_0} 2\pi \right)^2 + \dots \right) \\
&\approx \frac{1}{2} \frac{\Phi^2}{L J_0} \quad (\text{standard expression for mag. energy})
\end{aligned}$$

• Cooper pair box Hamiltonian operator

$$\hat{H} = E_C (\hat{N} - N_g)^2 - E_J \cos \hat{\delta}$$

$\frac{1}{2}(e^{i\hat{\delta}} + e^{-i\hat{\delta}})$

with commutation relation $[\hat{\delta}, \hat{N}] = i$ for conjugate variables δ and N .

Transformation between bases:

$$|\delta\rangle = \frac{1}{\sqrt{2\pi}} \sum_N e^{iN\delta} |N\rangle$$

number states

• properties of phase $\hat{\delta}$ and number \hat{N} operators

$$[\hat{\delta}, \hat{N}] = i \Rightarrow e^{\pm i\hat{\delta}} |N\rangle = |N \pm 1\rangle$$

number operator

$$\hat{N} |N\rangle = N |N\rangle$$

$$\sum_N |N\rangle\langle N| = \mathbb{1} \quad ; \quad \text{completeness}$$

$$\langle M | N \rangle = \delta_{M,N} \quad ; \quad \text{orthogonality}$$

Hamilton Operator of Cooper Pair Box in Charge Basis

$$\hat{H} = \sum_N \left(\underbrace{E_C (N - N_g)^2 |N\rangle\langle N|}_{\text{energy of charges on island}} - E_J/2 \underbrace{(|N\rangle\langle N+1| + |N+1\rangle\langle N|)}_{\text{energy to add or remove charges from island}} \right)$$

- solve time independent Schrödinger equation in discrete charge basis $|N\rangle$ to find energy eigenstates $|\psi\rangle$ of qubit

Show band diag. slides!

$$\hat{H} |\psi_m\rangle = E_m |\psi_m\rangle$$

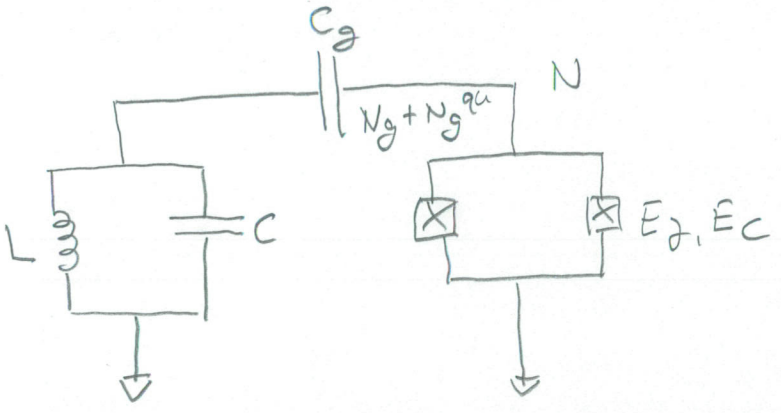
- equivalent Hamilton operator in phase basis δ (continuous & periodic)

$$\hat{N} = \frac{\hat{Q}}{2e} = -i\hbar \frac{1}{2e} \frac{\partial}{\partial \Phi} = -i \frac{\partial}{\partial \delta}$$

$$\hat{H} = E_C \left(-i \frac{\partial}{\partial \delta} - N_g \right)^2 - E_J \cos \delta$$

\Rightarrow exact solutions for $\hat{H} \psi_m(\delta) = E_m \psi_m(\delta)$ are Mathieu functions

Jaynes - Cummings Hamiltonian in Circuit QED



$$\hat{H} = \underbrace{\frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}}_{\text{H.O.}} + \frac{E_C}{2} \underbrace{(1 - 2(N_g + N_g^{qu}))}_{=0 \text{ at } N_g = \frac{1}{2}} \hat{\sigma}_z - \frac{E_J}{2} \hat{\sigma}_x$$

N_g^{qu} : quantum fluctuations of charge on capacitor C_g

$N_g = \frac{1}{2}$: consider charge degeneracy

quantum fluctuations of harmonic oscillator

$$\hat{H}_{HO} = \frac{1}{2} C \hat{V}^2 + \frac{1}{2} L \hat{I}^2$$

$$\hat{V} = \sqrt{\frac{\hbar \omega_r}{2C}} (\hat{a}^\dagger + \hat{a})$$

$$\Delta V^2 = \langle 0 | \hat{V}^2 | 0 \rangle - \underbrace{\langle 0 | \hat{V} | 0 \rangle^2}_{\text{mean voltage} = 0 \text{ for } |n\rangle = |0\rangle}$$

$$= \frac{\hbar \omega_r}{2C} \underbrace{\langle 0 | (\hat{a}^\dagger + \hat{a})^2 | 0 \rangle}_{=0} = \frac{\hbar \omega_r}{2C}$$

with quantum fluctuations of charge

$$N_g^{qu} = \frac{C_g}{2e} \hat{V}^{qu} = \frac{C_g}{2e} \sqrt{\frac{\hbar \omega_r}{2C}} (\hat{a}^\dagger + \hat{a})$$

Full Hamiltonian

(2)

- with change of basis $\hat{\sigma}_z \rightarrow \hat{\sigma}_x$ and $\hat{\sigma}_x \rightarrow -\hat{\sigma}_z$

$$\hat{H} = \hbar \omega_r (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \frac{E_c}{2} \frac{C_g}{2e} \sqrt{\frac{\hbar \omega_r}{2C}} (\hat{a}^\dagger + \hat{a}) \hat{\sigma}_x + \frac{E_J}{2} \hat{\sigma}_z$$

- and qubit raising and lowering operators $\hat{\sigma}^+$, $\hat{\sigma}^-$

$\hat{\sigma}_x = (\hat{\sigma}^+ + \hat{\sigma}^-)$ we find for the interaction

$$\frac{E_c}{2} \frac{C_g}{e} \sqrt{\frac{\hbar \omega_r}{2C}} (\hat{a}^\dagger \hat{\sigma}^+ + \hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+ + \hat{a} \hat{\sigma}^-)$$

rotating wave approximation (RWA)

- with $E_c = \frac{(2e)^2}{2C\Sigma}$ the full Hamiltonian reads

$$\hat{H} = \hbar \omega_r (\hat{a}^\dagger \hat{a}) + \hbar g (\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+) + \frac{E_J}{2} \hat{\sigma}_z$$

with coupling constant

$$\hbar g = \frac{C_g}{C\Sigma} 2e \sqrt{\frac{\hbar \omega_r}{2C}}$$

where $\frac{2g}{2\pi}$ is the vacuum Rabi frequency