Multiple Qubit States and Entanglement

How many different states can two classical or two quantum mechanical bits be in?

register of $n = 2$ classical bits:

BIT A	BIT B
0	0
0	1
1	0
1	1

2^n different states

note: - only one state is realized at any given time

register of $n = 2$ quantum bits

QUBIT $\|A\rangle$	QUBIT $\|B\rangle$
$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$

2^n basis states

BUT: - quantum register can be in any superposition of basis states

formal description of general state of $n=2$ quantum register

$$|\psi\rangle = |A\rangle \otimes |B\rangle = |AB\rangle \text{ (according to } 4^{th} \text{ postulate)}$$

e.g. $|A\rangle = \alpha_A |0\rangle + \beta_A |1\rangle$; $|B\rangle = \alpha_B |0\rangle + \beta_B |1\rangle$

$$|\psi\rangle = \alpha_A \alpha_B |00\rangle + \alpha_A \beta_B |01\rangle + \beta_A \alpha_B |10\rangle + \beta_A \beta_B |11\rangle$$

with $\sum_{ij} |\alpha_{ij}|^2 = 1$ (normalization condition for probabilities)

Information Content of Many Qubit States

register of n qubits:

- 2^n basis states
- general superposition state is described by 2^n complex coefficients

How would you best describe the state of $n=500$ qubits?
Is it at all possible?

Consider $n = 500$ qubits

- need $2^{500} = 3 \times 10^{150}$ coefficients
  - $\Rightarrow$ larger than number of atoms in universe
  - $\Rightarrow$ impossible to store information about state classically

This is why it is difficult to simulate QM on a classical computer. But it would be natural to simulate QM on a quantum computer.

Entangled Qubit States

<u>Definition</u>: An <u>entangled</u> state of a composite system is a state that <u>cannot</u> be written as a <u>product</u> state of the component systems.

product state: (example)

$$|\psi\rangle = |\psi_1 \psi_2\rangle \quad \text{with } |\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$
$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

entangled state: (example)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

How would you figure out, if this is an entangled state?

Does nature create such states?

How would you go about creating such a state?

$$\left.\begin{aligned} &\Rightarrow \alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \wedge \beta_1\beta_2 = \frac{1}{\sqrt{2}} \\ &\Rightarrow \alpha_1\beta_2 \neq 0 \wedge \alpha_2\beta_1 \neq 0 \end{aligned}\right\}$$ i.e. not a product state

<u>Questions</u>:

- How are such states created?
- What are their properties?

Correlations of Entangled States

measurement of individual qubit states in an entangled pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

What would you think is the result of a measurement of the state of both qubits?

- measure ground state $|0\rangle$ of first qubit (1)

$$\underset{\substack{\uparrow \\ \text{qubit}}}{P_1}\underset{\substack{\uparrow \\ \text{state}}}{(0)} = \langle\psi|\,(M_0 \otimes I)^\dagger \underbrace{(M_0 \otimes I)}_{\substack{\text{tensor products of individual} \\ \text{qubit measurement operators}}}|\psi\rangle = \frac{1}{2}$$

- post measurement state

$$|\psi'\rangle = \frac{(M_0 \otimes I)|\psi\rangle}{\sqrt{P_1(0)}} = |00\rangle$$

- measure ground state $|0\rangle$ of second qubit (2) given that first one was measured in state $|0\rangle$.

$$P_2(0) = \langle\psi'|\,(I \otimes M_0)^\dagger (I \otimes M_0)\,|\psi'\rangle = 1$$

$\Rightarrow$ The outcomes of the measurements of both qubit states are 100% correlated. Such correlations are impossible in a classical system (compare with Bell inequalities)

Entanglement as a New Resource

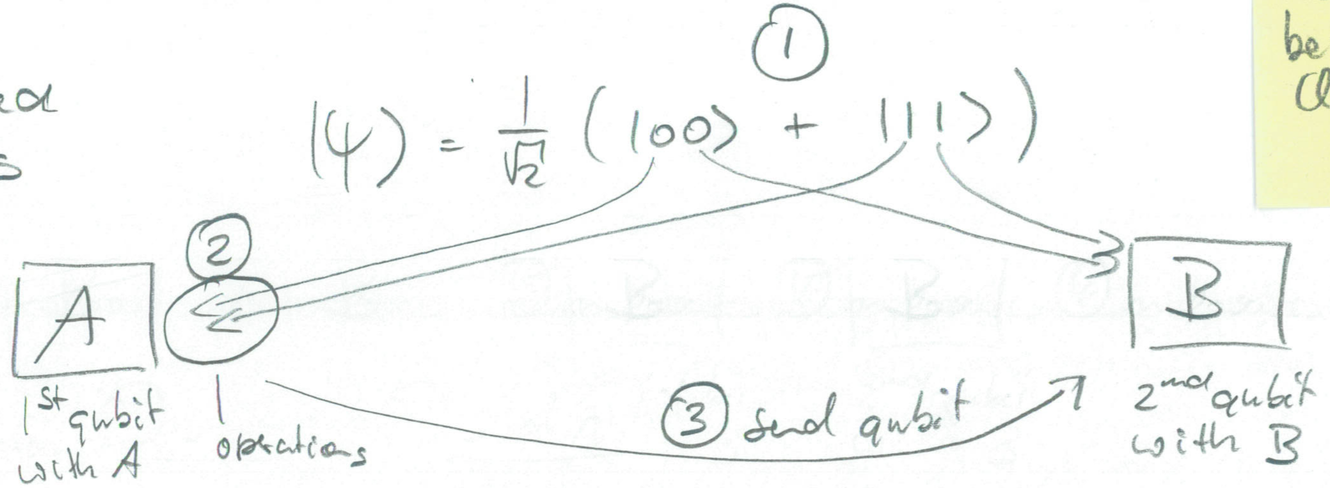
Transmit two bits of classical information by sending one qubit between two parties Alice and Bob: Super Dense Coding

How is this better than classical?

Protocol:

① share entangled pair of qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



original proposal by Wiesner and Bennett!

② Alice performs one of 4 local operations on her bit

$$\left.\begin{array}{l} I_1 \otimes I_2 \\ Z_1 \otimes I_2 \\ X_1 \otimes I_2 \\ iY_1 \otimes I_2 \end{array}\right\} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \left\{\begin{array}{ll} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \phi^+ & \longrightarrow 00 \\ \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = \phi^- & \longrightarrow 01 \\ \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right) = \psi^+ & \quad 0 \\ \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \psi^- & \end{array}\right.$$

2 classical bit

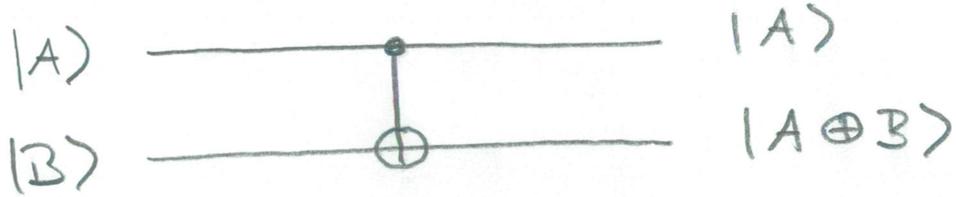
What about physical realization? → slides: realization with photons!

③ Alice sends qubit to Bob

④ Bob performs a measurement on both qubits and finds 4 outcomes

CNOT : A Universal 2-Qubit Logic Gate

Controlled NOT gate



CONTROL QUBIT

TARGET QUBIT

How would you realize a CNOT operation between two qubits?

truth table

INPUT	OUTPUT
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$

general

$|A, B\rangle$ $\quad$ $|A, A \oplus B\rangle$

addition mod 2

- is reversible (unitary)
- is universal
- can be realized using any two qubit interaction combined with single qubit manipulations

What is required on a physical level to realize conditional logic?
Do you know any types of interactions between quantum particles?

Universality

Any multi qubit logic gate can be composed of CNOT and single qubit gates (X, Y, Z, I).

Quantum Teleportation

Task: Transfer unknown quantum state $|\psi\rangle$ from <u>A</u>lice to <u>B</u>ob

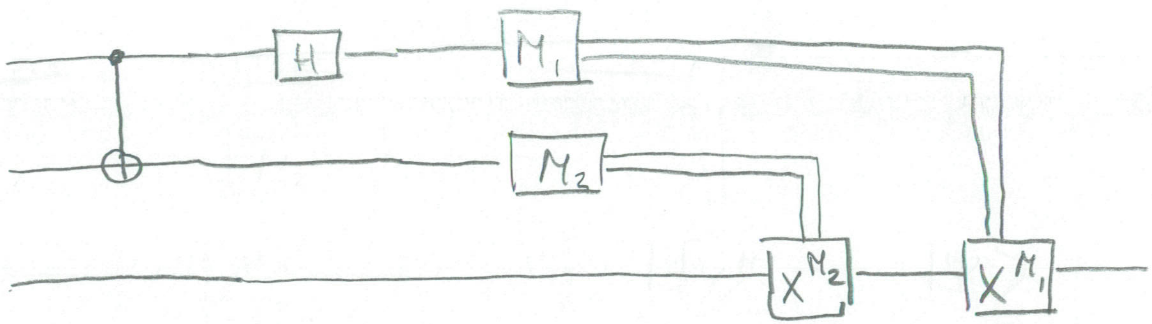
Resources: entangled pair of qubits & classical communication

How would you perform this task?

Circuit:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



$|\psi\rangle$

INPUT (ALICE) OUTPUT (BOB)

Steps: ① input ② CNOT ③ Hadamard ④ measurement ⑤ conditional operations ⑥ output

Note:
- A has no information about $|\psi\rangle$ (and cannot obtain it)
- state is always fully transferred

Teleportation Protocol

① Initial state $\quad (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

② $\xrightarrow{CNOT_{1,2}} \quad \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

③ $\xrightarrow{H_1} \quad \frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle$
$\quad + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$

What are the different measurement outcomes? With which prob. do they occur?

$$= \frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big)$$

④ measurement of qubit state

$M_1 \otimes M_2 \otimes I$

$P_{00} = P_{10} = P_{01} = P_{11} = \frac{1}{4}$

⑤ conditional qubit manipulations on post measurement state $|\psi'\rangle$

$$\left.\begin{aligned} |00\rangle &: \hat{I}|\psi'\rangle \\ |10\rangle &: \hat{Z}|\psi'\rangle \\ |01\rangle &: \hat{X}|\psi'\rangle \\ |11\rangle &: \hat{X}\hat{Z}|\psi'\rangle \end{aligned}\right\} = \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$

- requires transfer of two bits of classical information to <u>B</u>ob to perform local operations that recover the original state

<u>Note</u>: - state of one qubit transferred using one pair of entangled qubits and two bits of classical information

⇒ task cannot be performed classically

<u>Applications</u>:
- quantum error correction
- quantum gates
- quantum repeaters

original proposal : C. H. Bennett et al. Phys. Rev. Lett **70**, 1895 (1993)

first experimental implementation : D. Bouwmeester et al. Nature **390**, 575 (1997)

⇒ tested in different implementations using
- photons
- nuclear spins
- ions

⇒ hallmark quantum information processing experiment