Solutions problem set 2

1a) $|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right)$

Tak basis $\left(|0\rangle_A|0\rangle_B,\ |1\rangle_A|0\rangle_B,\ |0\rangle_A|1\rangle_B,\ |1\rangle_A|1\rangle_B\right)$

$$\Rightarrow |\psi^+\rangle\langle\psi^+| = \frac{1}{2}\left[\left(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\right)\left({}_B\langle 1|{}_A\langle 0| + {}_B\langle 0|{}_A\langle 1|\right)\right]$$

$$= \frac{1}{2}\left[|0\rangle_A|1\rangle_B\,{}_B\langle 1|{}_A\langle 0| + |0\rangle_A|1\rangle_B\,{}_B\langle 0|{}_A\langle 1| + |1\rangle_A|0\rangle_B\,{}_B\langle 1|{}_A\langle 0| + |1\rangle_A|0\rangle_B\,{}_B\langle 0|{}_A\langle 1|\right]$$

$$= \frac{1}{2}\begin{array}{c} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ {}_B\langle 0|{}_A\langle 0| \quad {}_B\langle 0|{}_A\langle 1| \quad {}_B\langle 1|{}_A\langle 0| \quad {}_B\langle 1|{}_A\langle 1| \end{array} \begin{array}{l} |0\rangle_A|0\rangle_B \\ |1\rangle_A|0\rangle_B \\ |0\rangle_A|1\rangle_B \\ |1\rangle_A|1\rangle_B \end{array} = \rho_{|\psi^+\rangle} = \rho_{AB}$$

Matrix components $\ell m \in V$ $\ell' m' \in \text{dual}(V)$

$$\Rightarrow \rho_A = \mathrm{tr}_B[\rho_{AB}] = \left(\sum_{i,j=0}^{1} c^{AB}_{ij}\, |i\rangle_A {}_A\langle j|\ \mathrm{tr}\left(|i\rangle_B {}_B\langle j|\right)\right)$$

$$= \sum_{j=0}^{1} {}_B\langle j|\rho_{AB}|j\rangle_B = \sum_{j=0}^{1}\sum_{\substack{\ell,m=0\\ \ell',m'=0}}^{1} {}_B\langle j|\, |\ell\rangle_A |m\rangle_B\, C^{AB}_{\ell m \ell' m'}\, {}_B\langle m'|{}_A\langle \ell'|\, |j\rangle_B$$

$$= \sum_{\substack{j,\ell,m=0\\ \ell',m'}}^{1} |\ell\rangle_A \underbrace{\langle j|m\rangle_B}_{\delta_{jm}} C^{AB}_{\ell m} \underbrace{{}_B\langle m'|j\rangle_B}_{\delta_{j,m'}} {}_A\langle \ell'|$$

$$= \sum_{\ell,\ell',j=0}^{1} |\ell\rangle_A {}_A\langle \ell'|\, C_{\ell j \ell' j} = \sum_{\ell,\ell'=0}^{1} |\ell\rangle_A {}_A\langle \ell'| \overbrace{\left(C_{\ell 0 \ell' 0} + C_{\ell 1 \ell' 1}\right)}^{\text{"}\mathrm{tr}_B(|i\rangle\langle i|)\text{"}}$$

$A\ B\ \bar{A}\ \bar{B}$ $\quad A\ B\ \bar{A}\ \bar{B}$

$\in V$ $\quad$ dual(V)

$$= \begin{cases} 0 & \text{otherwise} \\ \frac{1}{2} & 1010 \\ & 0101 \end{cases}$$

$$= \begin{array}{cc} {}_A\langle 0| & {}_A\langle 1| \end{array}$$
$$\begin{pmatrix} 1\left(0+\frac{1}{2}\right) & 1(0+0) \\ 1(0+0) & 1\left(\frac{1}{2}+0\right) \end{pmatrix} \begin{array}{l} |0\rangle_A \\ |1\rangle_A \end{array}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{fully mixed state}$$

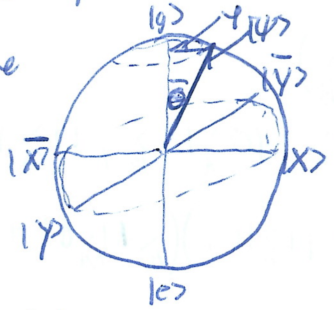
b) $\langle \hat{O} \rangle = \mathrm{tr}(\rho_A \hat{O}) \Rightarrow \langle \hat{O}_x \rangle = \frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\mathrm{tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{0}}$

$\langle \hat{O}_y \rangle = \underline{\underline{0}}$, likewise

$\langle \hat{O}_z \rangle = \frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \underline{\underline{0}}$

2
a)
c) Consider 1 qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos(\theta)|0\rangle + \sin(\theta)e^{i\varphi}|1\rangle$

Bloch sphere



$|X\rangle = (|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}$
$|\bar{X}\rangle = (|0\rangle - |1\rangle)\frac{1}{\sqrt{2}}$

$|y\rangle = (|0\rangle + i|1\rangle)\frac{1}{\sqrt{2}}$
$|\bar{y}\rangle = (|0\rangle - i|1\rangle)\frac{1}{\sqrt{2}}$

i) In the most ideal case 2 independent measurements are needed to determine θ, φ!

ii) 1 additional ~~case~~ measurement is needed if it is <u>not</u> given whether we have a <u>pure</u> state or a <u>mixed</u> state

iii) 1 additional measurement is needed if we don't know the <u>normalization</u> of our measurement or whether it is preserved

$\Rightarrow$ maximum <u>4</u> measurements.

b) Project $|\psi\rangle$ onto $|g\rangle, |e\rangle, |X\rangle, |y\rangle$ using a drive (e.g. $\Omega\sigma_x$, $\Omega = \varepsilon e^{i\varphi}$)

gives ~~contrast~~ ~~prob~~ z-component (θ)

normalizes (with $|g\rangle$)

checks if mixed

"gives φ"

Since we can only measure e.g. the population in ground state $|g\rangle$ follow the protocol

i) Prepare $|\psi\rangle \rightarrow$ measure population in $|g\rangle \Rightarrow P_{|g\rangle}$
ii) " " $\rightarrow$ do π rotation around $|x\rangle \Rightarrow P_{|e\rangle}$
iii) " " $\rightarrow$ do $-\pi/2$ rotation around $|x\rangle \Rightarrow P_{|y\rangle}$
iv) " " $\rightarrow$ do $\pi/2$ " " $|y\rangle \Rightarrow P_{|x\rangle}$

do N times

Define Stokes parameter

$$S_0 = P_{|g\rangle} + P_{|e\rangle} = P_{|g\rangle} + P_{|e\rangle} = 1$$
$$S_1 = P_{|x\rangle} - P_{|\bar{x}\rangle} = 2P_{|x\rangle} - 1$$
$$S_2 = P_{|y\rangle} - P_{|\bar{y}\rangle} = 2P_{|y\rangle} - 1$$
$$S_3 = P_{|g\rangle} - P_{|e\rangle} = 2P_{|g\rangle} - 1$$

$S_0 = 1$ normalized

$$\Rightarrow \rho_{|\psi\rangle\langle\psi|} = \sum_{i=1}^{4} S_i \sigma_i$$

$\mathbb{1}$ and Pauli Matrix

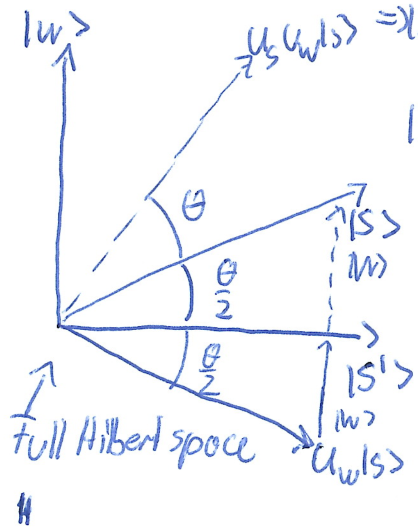
d) <u>4^n</u>

3) a) $N = 2^n \Rightarrow n=4$

b) $|x\rangle$ all states in Hilbertspace

$|w\rangle$ searched state

i) $|s\rangle = \frac{1}{\sqrt{N}}\sum_{x=1}^{N} |x\rangle$ start state $\Rightarrow$ maximally unknown state $\Leftrightarrow$ no other state is more probable than another $\Rightarrow$ unsorted



$\Rightarrow |s'\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq w}|x\rangle$ start state without $|w\rangle$

$|s'\rangle$ orthogonal to $|w\rangle$

ii) Define "oracle" $U_w = \mathbb{1} - 2|w\rangle\langle w| \Rightarrow \underbrace{\begin{cases} |x\rangle & |x\rangle \neq |w\rangle \\ -|w\rangle & |x\rangle = |w\rangle \end{cases}}_{U_w|x\rangle}$

iii) Apply $U_w|s\rangle \Rightarrow$ mirror $|s\rangle$ at $|s'\rangle$ (subtract $|w\rangle$ component twice)

iv) realize $\langle s'|s\rangle =$ ~~$\sum_{x=1}^{N}\sum_{x'=1}^{N}\langle x'|x\rangle \ldots$~~

$= \langle s'|\left(\sum_{x\neq w}|x\rangle + |w\rangle\right)\cdot\frac{1}{\sqrt{N}}$

$= \langle s'|\left(\frac{\sqrt{N-1}}{\sqrt{N}}|s'\rangle + \frac{1}{\sqrt{N}}|w\rangle\right) = \frac{\sqrt{N-1}}{\sqrt{N}}\underbrace{\langle s'|s'\rangle}_{1} + \frac{1}{\sqrt{N}}\underbrace{\langle s'|w\rangle}_{0}$

$= \sqrt{\frac{N-1}{N}}$

$\Rightarrow \langle s'|s\rangle = 1\cdot 1\cdot\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{N-1}{N}}$

$|s'| = |s| = 1$

$\Rightarrow \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 = \frac{N-1}{N} + \sin^2\frac{\theta}{2}$

$= 1 - \frac{1}{N} + \sin^2\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2} = \sqrt{\frac{1}{N}}$

$\Rightarrow \boxed{\arctan\frac{1}{\sqrt{N}} = \frac{\theta}{2}}$

v) Define + apply $U_s = 2|s\rangle\langle s| - \mathbb{1}$ (reflect state through $|s\rangle$)

$\Rightarrow$ Every application of $U_sU_w|g-1\rangle = |g\rangle$, with $|0\rangle = |s\rangle$ rotates state by θ

grover steps

vi) $\Rightarrow |\langle w|g\rangle|^2 = \sin^2\left(\left(g+\frac{1}{2}\right)\theta\right) = 1$ for $\left(g+\frac{1}{2}\right)\theta = \frac{\pi}{2} \Rightarrow g\frac{2}{\sqrt{N}} + \frac{1}{\sqrt{N}} = \frac{\pi}{2} \Rightarrow \boxed{g \approx \frac{\pi\sqrt{N}}{4}}$

$\theta \approx \frac{2}{\sqrt{N}}$ for $N \gg 1$; $\frac{1}{\sqrt{N}} \approx 0$

$\Rightarrow$ c) $g \approx 3$