Solutions exercise "4" or "5"

1a) $H_{CPB} = \sum_n E_c(\hat{n}-n_g)^2 |n\rangle\langle n| - \frac{E_J}{2}(|n\rangle\langle n+1| + |n+1\rangle\langle n|)$

$$\overset{n\in(0,1)}{=} E_c(-n_g)^2|0\rangle\langle 0| + E_c(1-n_g)^2|1\rangle\langle 1| - \frac{E_J}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$= \begin{pmatrix} E_c(-n_g)^2 & -E_J/2 \\ -E_J/2 & E_c(1-n_g)^2 \end{pmatrix} \underset{n_g = n_g^0 + \underset{\substack{V_0\cos(\omega t)\\ \text{external drive}}}{V(t)}}{=} \begin{pmatrix} E_c(-n_g^0-V)^2 & -E_J/2 \\ -E_J/2 & E_c(1-n_g^0-V)^2 \end{pmatrix} \overset{V\ll n_g^0}{\approx} \begin{pmatrix} E_c(n_g^{0\,2} - 2Vn_g^0) & -E_J/2 \\ -E_J/2 & E_c((1-n_g^0)^2 - 2V(1-n_g^0)) \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{E_c n_g^{0\,2}}_{E_1} & -E_J/2 \\ -E_J/2 & \underbrace{E_c(1-n_g^0)^2}_{E_2} \end{pmatrix} - 2V(t)E_c \begin{pmatrix} n_g^0 & \\ & (1-n_g^0) \end{pmatrix}$$ (split off time dependent Hamiltonian)

diagonalize in static Eigenbasis (Mathematica)

$$\underset{E_2 - E_1 = \Delta = E_c(1-2n_g^0)}{=} \begin{pmatrix} -\frac{1}{2}\sqrt{E_J^2+\Delta^2} & 0 \\ 0 & \underbrace{\frac{1}{2}\sqrt{E_J^2+\Delta^2}}_{\hbar\omega_0/2} \end{pmatrix} - 2V(t)E_c\, U^{-1} \begin{pmatrix} n_g^0 & 0 \\ 0 & 1-n_g^0 \end{pmatrix} U$$

$$U = \begin{pmatrix} \frac{\Delta - \sqrt{E_J^2+\Delta^2}}{E_J} & \frac{-\Delta + \sqrt{E_J^2+\Delta^2}}{E_J} \\ 1 & 1 \end{pmatrix}, \quad \underbrace{}_{EV_1} \quad \underbrace{}_{EV_2}$$

$$= -\frac{\hbar\omega_0}{2}\hat{\sigma}_z - \underbrace{2VE_c}_{A\cos(\omega t)} \begin{pmatrix} -1+2n_g^0 - \frac{\Delta}{\sqrt{E_J^2+\Delta^2}} & -1 - \frac{\Delta}{\sqrt{E_J^2+\Delta^2}} \\ -1 + \frac{\Delta}{\sqrt{E_J^2+\Delta^2}} & -1+2n_g^0 + \frac{\Delta}{\sqrt{E_J^2+\Delta^2}} \end{pmatrix}$$

$n_g^0 = 0.5$
$\Rightarrow \Delta = 0$

$$= -\frac{\hbar\omega_0}{2}\hat{\sigma}_z - A\cos(\omega t)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\frac{\hbar\overset{=\Omega}{\omega_0}}{2}\hat{\sigma}_z + A\cos(\omega t)\hat{\sigma}_x$$

b) i) $$\frac{H}{\hbar} = \vec{m}(t)\cdot\vec{\sigma} \underset{\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}\text{-basis}}{=} \begin{pmatrix} A\cos(\omega t) \\ 0 \\ \Omega/2 \end{pmatrix} \underset{\substack{\text{Decompose in co}\\ \text{and counter rotating}\\ \text{components}}}{=} \frac{1}{2}\begin{pmatrix} A\cos(\omega t) \\ A\sin(\omega t) \\ \Omega/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A\cos(\omega t) \\ -A\sin(\omega t) \\ \Omega/2 \end{pmatrix}$$

$$\Rightarrow H = -\frac{\hbar\Omega}{2}\sigma_z + \frac{A}{2}\underbrace{(\cos(\omega t)\sigma_x - \sin(\omega t)\sigma_y)}_{-} + \frac{A}{2}\underbrace{(\cos(\omega t)\sigma_x + \sin(\omega t)\sigma_y)}_{+}$$

ii) $\hat{H}|\psi\rangle = i\hbar|\dot\psi\rangle \Rightarrow UHU^\dagger\underbrace{|\phi\rangle}_{|\psi\rangle} = U i\hbar \underbrace{(U^\dagger|\phi\rangle)^{\cdot}}_{\dot\psi} = i\hbar(U\dot U^\dagger|\phi\rangle + UU^\dagger|\dot\phi\rangle)$

$$\Rightarrow \underbrace{\left(UHU^\dagger - i\hbar U\dot U^\dagger\right)}_{\tilde H}|\phi\rangle = i\hbar|\dot\phi\rangle$$

= correct "Basis" Transformation of Hamiltonian for unitary Transformations

$\hat\sigma$ Lie algebra of SU(2) rotations

iii) $U = \begin{pmatrix} e^{+i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} = \exp\left(+i\frac{\omega t}{2}\hat\sigma_z\right)$ for rotating with $\underline{\omega t}$ around z-axis.

because $\vec S = \frac{1}{2}$-particle, $\underline{SU(2)}$!

$U\hat\sigma_x U^\dagger = \cos(\omega t)\hat\sigma_x - \sin(\omega t)\hat\sigma_y$

$U\hat\sigma_y U^\dagger = +\sin(\omega t)\hat\sigma_x + \cos(\omega t)\hat\sigma_y$

$U\hat\sigma_z U^\dagger = \hat\sigma_z$

$U\dot U^\dagger = i\frac{\omega}{2}\sigma_z$

IV) $$\tilde H = -\frac{\hbar\omega_0}{2}\hat\sigma_z - i\hbar(+i)\frac{\omega}{2}\hat\sigma_z + \frac{A}{2}\Big[\cos\omega t\overbrace{(\cos\omega t\,\sigma_x - \sin\,\hat\sigma_y)}^{U\sigma_x U^\dagger} - \sin\omega t\overbrace{(+\sin\omega t\,\sigma_x + \cos\omega t\,\sigma_y)}^{U\sigma_y U^\dagger} + \cos\omega t\overbrace{(\cos\omega t\,\sigma_x - \sin\,\sigma_y)}^{U\sigma_x U^\dagger} + \sin\omega t\overbrace{(+\sin\omega t\,\sigma_x + \cos\omega t\,\sigma_y)}^{U\sigma_y U^\dagger}\Big]$$

$$\overset{?}{=} -\frac{\hbar\overbrace{(\omega_0-\omega)}^{\Delta}}{2}\sigma_z + \frac{A}{2}\Big[\sigma_x + \underbrace{\cos(2\omega t)\sigma_x - \sin(2\omega t)\sigma_y}_{\text{rotate too fast} \Rightarrow \text{neglect}}\Big]$$

trigonometric identities

(c) Rotating wave approximation

$$\Rightarrow \tilde H_{RWA} = -\frac{\hbar\Delta}{2}\hat\sigma_z + \frac{A}{2}\hat\sigma_x$$

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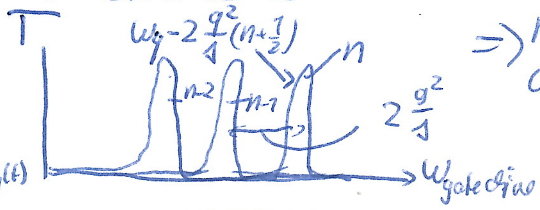
2) a) measure transmission of resonator $\Rightarrow$ resonance frequency of resonator $\tilde\omega_r = \omega_r + \frac{g^2}{\Delta}\langle\sigma_z\rangle$

$= \omega_r \pm \frac{g^2}{\Delta}$ {depending of qubit in $|e\rangle$ or $|g\rangle$}

b) $\tilde\omega_q = \omega_q - 2\frac{g^2}{\Delta}\left(\langle a^\dagger a\rangle + \frac{1}{2}\right) = \omega_q - 2\frac{g^2}{\Delta}\left(n + \frac{1}{2}\right)$

n: photon number in resonator

c) continuously measure transmission of resonator at $\omega_r + \frac{g^2}{\Delta}$ (~~if qubit~~ $\Rightarrow 0$ if qubit in $|g\rangle$, (1) if qubit in $|e\rangle$ $\Rightarrow$ change frequency on $\Omega_q(t)$



$\Rightarrow$ measure resonance at qubit frequency